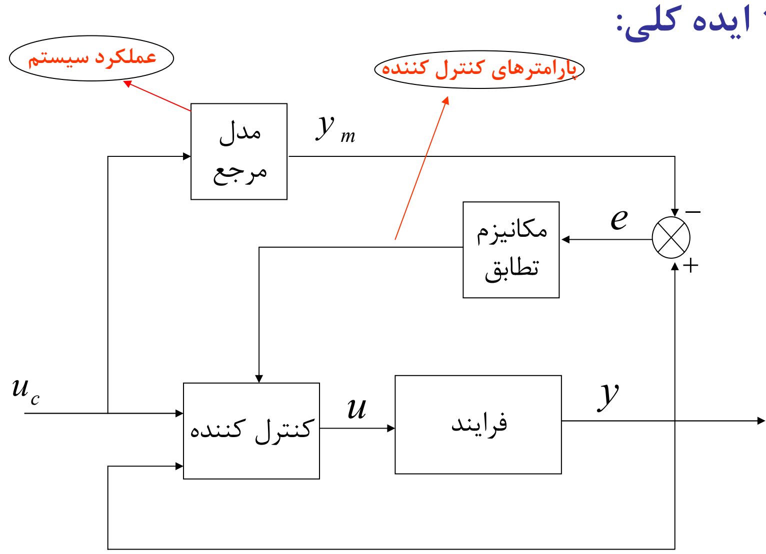
سیستم های تطبیقی مدل مرجع
(Model Reference Adaptive Systems)

علی خاکی صدیق

گروه کنترل

آذر ۱۳۸۵

❖ ایده کلی:



✓ ۲ حلقه کنترلی:

- حلقه داخلی
- حلقه خارجی

✓ مساله اصلی : مکانیزم تطابق یا Adaptation Mechanism

↓

Stable MRAS

↓

Robust MRAS

- **سه روند اساسی تحلیل و طراحی MRAS:**

✓ روش گرادیان

✓ روش لیاپانوف

✓ روش Passivity

روشهای مبتنی بر نظریه پایداری

- STR یا MRAS ؟
- نخستین کاربردها و تجربیات

❖ روش گرادیان (MIT Rule)

- ایده کلی: تغییر پارامترهای رگلاتور به گونه ای که خطای بین خروجی فرایند و خروجی مدل مرجع به صفر میل کند.

- معیار عملکرد:

تغییر پارامترها در جهت گرادیان منفی

$$J(\theta) = \frac{1}{2} e^2$$

$$\Rightarrow \frac{d\theta}{dt} = -\gamma \frac{\partial J}{\partial \theta} = -\gamma e \frac{\partial e}{\partial \theta} \quad \text{(MIT Rule)}$$

مشتقات حساسیت

EXAMPLE 5.1 Adaptation of a feedforward gain

Consider the problem of adjusting a feedforward gain. In this problem it is assumed that the process is linear with the transfer function $kG(s)$, where $G(s)$ is known and k is an unknown parameter. The underlying design problem is to find a feedforward controller that gives a system with the transfer function $G_m(s) = k_0 G(s)$, where k_0 is a given constant. With the feedforward controller

$$u = \theta u_c$$

where u is the control signal and u_c the command signal, the transfer function from command signal to the output becomes $\theta k G(s)$. This transfer function is equal to $G_m(s)$ if the parameter θ is chosen to be

$$\theta = \frac{k_0}{k}$$

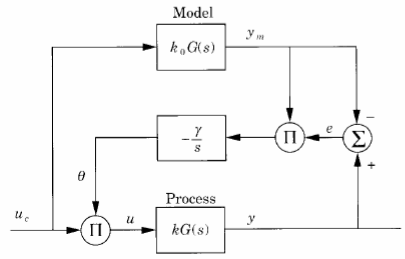


Figure 5.2 Block diagram of an MRAS for adjustment of a feedforward gain based on the MIT rule.

We will now use the MIT rule to obtain a method for adjusting the parameter θ when k is not known. The error is

$$e = y - y_m = kG(p)\theta u_c - k_0 G(p) u_c$$

where u_c is the command signal, y_m is the model output, y is the process output, θ is the adjustable parameter, and $p = d/dt$ is the differential operator. The sensitivity derivative is given by

$$\frac{\partial e}{\partial \theta} = kG(p)u_c = \frac{k}{k_0} y_m$$

The MIT rule then gives the following adaptation law:

$$\frac{d\theta}{dt} = -\gamma' \frac{k}{k_0} y_m e = -\gamma y_m e \tag{5.5}$$

where $\gamma = \gamma' k / k_0$ has been introduced instead of γ'. Notice that to have the correct sign of γ, it is necessary to know the sign of k. Equation (5.5) gives the law for adjusting the parameter. A block diagram of the system is shown in Fig. 5.2.

The properties of the system can be illustrated by simulation. Figure 5.3 shows a simulation when the system has the transfer function

$$G(s) = \frac{1}{s+1}$$

The input u_c is a sinusoid with frequency 1 rad/s, and the parameter values are $k = 1$ and $k_0 = 2$. Figure 5.3 shows that the parameter converges toward the correct value reasonably fast when the adaptation gain is $\gamma = 1$ and that the process output approaches the model output. Figure 5.3 also shows that

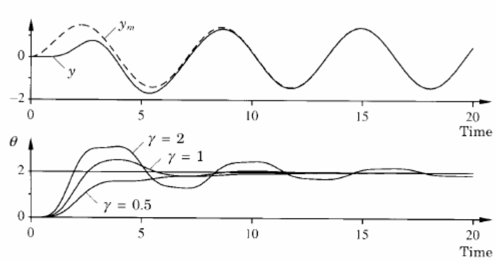


Figure 5.3 Simulation of an MRAS for adjusting a feedforward gain. The process (solid line) and the model (dashed line) outputs are shown in the upper graph for $\gamma = 1$. The controller parameter is shown in the lower graph when the adaptation gain γ has the values 0.5, 1, and 2.

the convergence rate depends on the adaptation gain. It is thus important to know a reasonable value of this parameter. Intuitively, we may expect that parameters converge slowly for small γ and that the convergence rate increases with γ. Simulation experiments indicate that this is true for small values of γ but also that the behavior is quite unpredictable for large γ. □

EXAMPLE 5.2 MRAS for a first-order system

Consider a system described by the model

$$\frac{dy}{dt} = -ay + bu \tag{5.6}$$

where u is the control variable and y is the measured output. Assume that we want to obtain a closed-loop system described by

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

Let the controller be given by

$$u(t) = \theta_1 u_c(t) - \theta_2 y(t) \tag{5.7}$$

The controller has two parameters. If they are chosen to be

$$\begin{aligned} \theta_1 &= \theta_1^0 = \frac{b_m}{b} \\ \theta_2 &= \theta_2^0 = \frac{a_m - a}{b} \end{aligned} \tag{5.8}$$

the input-output relations of the system and the model are the same. This is called perfect model-following.

To apply the MIT rule, introduce the error

$$e = y - y_m$$

where y denotes the output of the closed-loop system. It follows from Eqs. (5.6) and (5.7) that

$$y = \frac{b\theta_1}{p + a + b\theta_2} u_c$$

where $p = d/dt$ is the differential operator. The notation used is discussed in Section 1.5. The sensitivity derivatives are obtained by taking partial derivatives with respect to the controller parameters θ_1 and θ_2:

$$\frac{\partial e}{\partial \theta_1} = \frac{b}{p + a + b\theta_2} u_c$$

$$\frac{\partial e}{\partial \theta_2} = -\frac{b^2\theta_1}{(p + a + b\theta_2)^2} u_c = -\frac{b}{p + a + b\theta_2} y$$

These formulas cannot be used directly because the process parameters a and b are not known. Approximations are therefore required. One possible approximation is based on the observation that $p + a + b\theta_2^0 = p + a_m$ when the

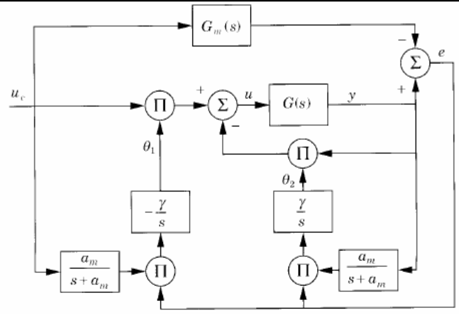


Figure 5.4 Block diagram of a model-reference controller for a first-order process.

parameters give perfect model-following. We will therefore use the approximation

$$p + a + b\theta_2 \approx p + a_m$$

which will be reasonable when parameters are close to their correct values. With this approximation we get the following equations for updating the controller parameters:

$$\begin{aligned} \frac{d\theta_1}{dt} &= -\gamma \left(\frac{a_m}{p + a_m} u_c \right) e \\ \frac{d\theta_2}{dt} &= \gamma \left(\frac{a_m}{p + a_m} y \right) e \end{aligned} \tag{5.9}$$

In these equations we have combined parameters b and a_m with the adaptation gain γ', since they appear as the product $\gamma' b / a_m$. The sign of parameter b must be known to have the correct sign of γ. Notice that the filter has also been normalized so that its steady-state gain is unity.

The adaptive controller is a dynamical system with five state variables that can be chosen to be the model output, the parameters, and the sensitivity derivatives. A block diagram of the system is shown in Fig. 5.4. The behavior of the system is now illustrated by a simulation. The parameters are chosen to be $a = 1$, $b = 0.5$, and $a_m = b_m = 2$, the input signal is a square wave with amplitude 1, and $\gamma = 1$. Figure 5.5 shows the results of a simulation. Figure 5.6 shows the parameter estimates for different values of the adaptation gain γ. Notice that the parameters change most when the command signal changes

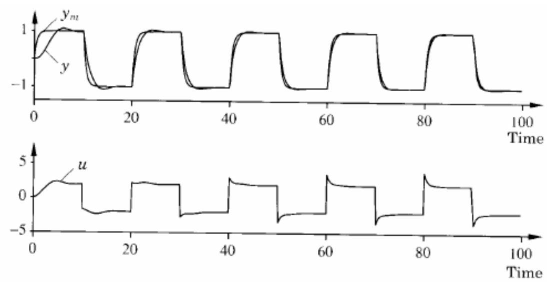


Figure 5.5 Simulation of the system in Example 5.2 using an MRAS. The parameter values are $a = 1$, $b = 0.5$, $a_m = b_m = 2$, and $\gamma = 1$.

and that the parameters converge very slowly. For $\gamma = 1$, the value used in Fig. 5.5, the parameters have the values $\theta_1 = 3.2$ and $\theta_2 = 1.2$ at time $t = 100$. These values are far from the correct values $\theta_1^0 = 4$ and $\theta_2^0 = 2$. However, the parameters will converge to the true values with increasing time. The convergence rate increases with increasing γ, as is shown in Fig. 5.6.

The fact that the control is quite good even at time $t = 10$ is a reflection of the fact that the parameter estimates are related to each other in a very special way, although they are quite far from their true values. This is illustrated in Fig. 5.7, where parameter θ_2 is plotted as a function of θ_1 for a simulation with a duration of 500 time units. Figure 5.7 shows that parameters do indeed

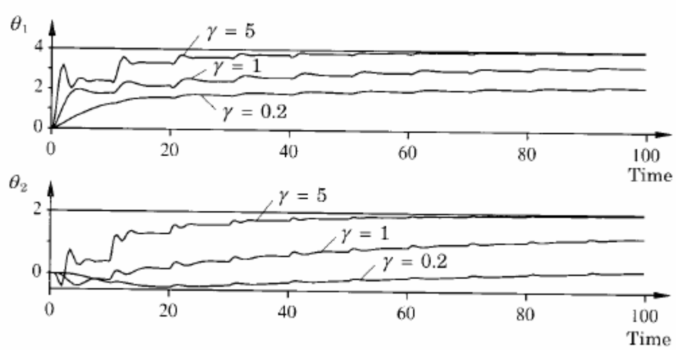


Figure 5.6 Controller parameters θ_1 and θ_2 for the system in Example 5.2 when $\gamma = 0.2$, 1 and 5.

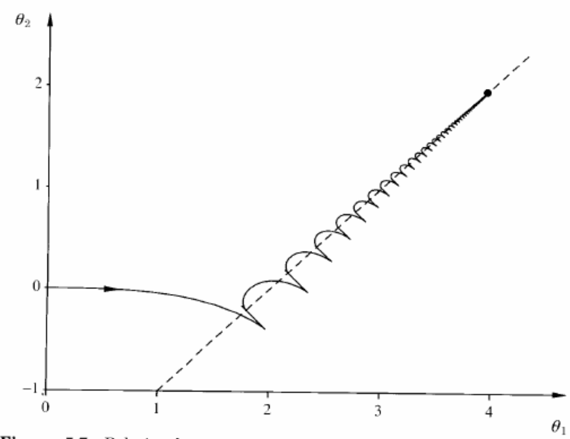


Figure 5.7 Relation between controller parameters θ_1 and θ_2 when the system in Example 5.2 is simulated for 500 time units. The dashed line shows the line $\theta_2 = \theta_1 - a/b$. The dot indicates the convergence point.

approach their correct values as time increases. The parameter estimates quickly approach the line $\theta_2 = \theta_1 - a/b$. This line represents parameter values such that the closed-loop system has correct steady-state gain. □

- **سیستم های خطی کلی**

✓ مدل سیستم

$$Ay = Bu$$

✓ مدل مرجع

$$A_m y_m = B_m u_c$$

✓ رگلاتور

$$Ru = Tu_c - Sy$$

✓ سیستم حلقه بسته

$$y = \frac{BT}{AR+BS} u_c , u = \frac{AT}{AR+BS} u_c$$

✓ خطا

$$e = y - y_m$$

- برای به دست آوردن قانون تنظیم پارامتر باید مشتقات حساسیت (مشتقات نسبی خطا نسبت به پارامترهای رگلاتور) را تعیین کرد:

مشتقات حساسیت: $t_i, s_i, r_i, \to T, R, S$

$$\frac{\partial e}{\partial r_i} = -\frac{BTAp^{k-i}}{(AR+BS)^2}u_c = -\frac{Bp^{k-i}}{AR+BS}u \qquad i = 1,\cdots,k, k = \deg R$$

$$\frac{\partial e}{\partial s_i} = -\frac{BTBp^{l-i}}{(AR+BS)^2}u_c = -\frac{Bp^{l-i}}{AR+BS}y \qquad i = 1,\cdots,l, l = \deg S$$

$$\frac{\partial e}{\partial t_i} = \frac{Bp^{m-i}}{AR+BS}u_c \qquad i = 1,\cdots,m, m = \deg T$$

سیستم نامعلوم لذا محاسبه مشتقات ؟ لذا استفاده از تقریب

یک تقریب:

$$AR + BS \approx A_m B^+$$

$$\Rightarrow \frac{\partial e}{\partial r_i} \approx -\frac{B^- p^{k-i}}{A_m}u, \frac{\partial e}{\partial s_i} \approx -\frac{B^- p^{l-i}}{A_m}y, \frac{\partial e}{\partial t_i} \approx \frac{B^- p^{m-i}}{A_m}u_c$$

با وجود B^- باز هم غیر قابل تحقق است. برای سیستم های می نیمم فاز تمام صفرها حذف می شوند و $B^- = b_0$ تنها دانستن علامت b_0 کافی است. بهره b_0 را می توان در بهره تطابق جذب کرد:

$$\boxed{\frac{dr_i}{dt} = \gamma e\frac{p^{k-i}}{A_m}u, \frac{ds_i}{dt} = \gamma e\frac{p^{l-i}}{A_m}y, \frac{dt_i}{dt} = -\gamma e\frac{p^{m-i}}{A_m}u_c}$$

فرضیات: سیستم می نیمم فاز با علامت معلوم بهره. از RLS می توان برای تخمین پارامترهای سیستم استفاده کرد.

• مثال: MRAS با روش گرادیان

فرایند: $(b_0, a_1, a_2$ Unknown$)$

$$g(s) = \frac{b_0}{s^2 + a_1 s + a_2}$$

مدل مرجع:

$$g_m(s) = \frac{b_m}{s^2 + a_{1m} s + a_{2m}}$$

کنترل:

$$u = f u_c - q_0 \dot{y} - q_1 y$$

سیستم حلقه بسته:

$$y = \frac{f b_0}{s^2 + (a_1 + b_0 q_0)s + (a_2 + b_0 q_1)} u_c$$

پارامترهای کنترل:

$$f, q_0, q_1$$

ضرایب حساسیت:

$$\frac{\partial e}{\partial f} = \frac{b_0}{s^2 + (a_1 + b_0 q_0)s + (a_2 + b_0 q_1)} u_c$$

$$\frac{\partial e}{\partial q_0} = -\frac{f b_0 (b_0 s)}{[s^2 + (a_1 + b_0 q_0)s + (a_2 + b_0 q_1)]^2} u_c$$

$$\frac{\partial e}{\partial q_1} = -\frac{f b_0 (b_0)}{[s^2 + (a_1 + b_0 q_0)s + (a_2 + b_0 q_1)]^2} u_c$$

قاعده MIT با تقریب:

$$\frac{df}{dt} = -\gamma_1 e \frac{\partial e}{\partial f} = -\gamma_1 e \frac{b_0}{s^2 + (a_1 + b_0 q_0)s + (a_2 + b_0 q_1)} u_c \approx -\gamma_1 e \frac{b_0}{b_m} y_m$$

بنابراین:

$$\boxed{\frac{df}{dt} = -\bar{\gamma}_1 e y_m}$$

هم چنین:

$$\frac{dq_0}{dt}=-\gamma_2 e\frac{\partial e}{\partial q_0}=-\gamma_2 e\frac{-fb_0^2 s}{[s^2+(a_1+b_0q_0)s+(a_2+b_0q_1)]^2}u_c$$

$$=\gamma_2 e\frac{b_0 s}{s^2+(a_1+b_0q_0)s+(a_2+b_0q_1)}y$$

بنابراین:

$$\approx \gamma_2 e\frac{b_0 s}{s^2+a_{m1}s+a_{m2}}y$$

$$\boxed{\frac{dq_0}{dt}=\overline{\gamma}_2 e\frac{s}{s^2+a_{m1}s+a_{m2}}y}$$

هم چنین:

$$\frac{dq_1}{dt}=-\gamma_3 e\frac{\partial e}{\partial q_1}=-\gamma_3 e\frac{-fb_0^2}{[s^2+(a_1+b_0q_0)s+(a_2+b_0q_1)]^2}u_c$$

$$\approx \overline{\gamma}_3 e\frac{1}{s^2+a_{m1}s+a_{m2}}y$$

بنابراین:

$$\boxed{\frac{dq_1}{dt}=\overline{\gamma}_3 e\frac{1}{s^2+a_{m1}s+a_{m2}}y}$$

- خطای سیستم تطبیقی و همگرایی پارامترها

مثال

System: $y = u$

Model Reference: $y_m = \theta^0 u_c$

Controller: $u = \theta u_c$

Error: $e = (\theta - \theta^0) u_c$

MIT Rule:

$$\frac{d\theta}{dt} = -\gamma u_c^2 (\theta - \theta^0)$$

$$\Rightarrow \theta(t) = \theta^0 + (\theta(0) - \theta^0) e^{-\gamma I_t}$$

$$I_t = \int_0^t u_c^2 d\tau$$

Hence, $e(t) = u_c(t)(\theta(0) - \theta^0) e^{-\gamma I_t} \to 0$

- تعیین و تحلیل بهره تطابق

برای تطابق بهره با تابع تبدیل معلوم و پایدار داریم:

$$\frac{d\theta}{dt} = -\gamma y_m (kG(p)u - y_m)$$

$$= -\gamma y_m (kG(p)\theta u_c) + \gamma y_m^2$$

لذا معادله پارامتر سیستم عبارت است از

$$\frac{d\theta}{dt} + \gamma y_m (kG(p)\theta u_c) = \gamma y_m^2$$

که یک معادله دیفرانسیل تغییر پذیر با زمان است و تعیین اثر بهره تطابق در حالت کلی بسیار دشوار است.

طرح آزمایش:

ثابت θ, u_c

$\Downarrow$

تطابق خاموش و با رسیدن به حالت ماندگار دوباره راه اندازی می شود.

$\Downarrow$

$$\underbrace{\frac{d\theta}{dt} + \gamma y_m^o u_c^o (kG(p)\theta) = \gamma (y_m^o)^2}_{\text{معادله دیفرانسیل خطی تغییرناپذیر با زمان}}$$

$\Downarrow$

پایداری

$\Downarrow$

$$s + \gamma y_m^o u_c^o kG(s) = 0$$

$$s + \mu G(s) = 0$$

تعیین رفتار صفرهای معادله با رسم مکان ریشه به ازا تغییرات این پارامتر امکان پذیر است

$\Downarrow$

تعمیم به حالتی که u_c به کندی نسبت به دینامیک $G(s)$ تغییر می کند

• نکته مهم: اگر تفاضل قطب-صفر سیستم بیش از یک باشد و

$$\mu \rightarrow \gamma \text{ or } u_c^o$$

خیلی بزرگ باشد سیستم ناپایدار خواهد شد.

EXAMPLE 5.4 Choice of adaptation gain

Consider the system in Example 5.1 with $G(s) = 1/(s+1)$, $k = 1$, and $k_0 = 2$. Assume that the reference signal has unit amplitude. Equation (5.13) then becomes

$$s^2 + s + \mu = s^2 + s + \gamma y_m^o u_c^o k = 0$$

A reasonable choice is to make $\gamma y_m^o u_c^o k = 1$. If we disregard the transients, the average value of $y_m u_c$ is 2. This gives $\gamma = 0.5$, which is the value used in one of the simulations in Fig. 5.3. □

EXAMPLE 5.5 Stability depends on the signal amplitudes

Consider the system in Example 5.1. Let the transfer function G be given by

$$G(s) = \frac{1}{s^2 + a_1 s + a_2}$$

Equation (5.13) then becomes

$$s^3 + a_1 s^2 + a_2 s + \mu = 0$$

where $\mu = \gamma y_m^o u_c^o k$. The equation has all its roots in the left half-plane if

$$\gamma y_m^o u_c^o k < a_1 a_2 \qquad (5.15)$$

Since this inequality involves the magnitude of the command signal, it may happen that the equilibrium solution corresponding to one command signal is stable and the solution corresponding to another command signal is unstable. This is illustrated by the simulation results shown in Fig. 5.8, where parameters are chosen so that $k = a_1 = a_2 = 1$. In the simulation the adaptation rate γ was adjusted to give a good response when u_c is a square wave with unit amplitude. In this case we have $u_c^o = y_m^o = 1$, and inequality (5.15) gives the stability condition $\gamma < 1$. A reasonable value of γ is $\gamma = 0.1$, which was used in the simulation. Figure 5.8 shows clearly that the convergence rate depends on the magnitude of the command signal. Notice that the solution is unstable when the amplitude of u_c is 3.5. The approximate model predicts instability for u_c larger than 3.16. Also notice that the response is intolerably slow for low amplitudes of u_c. □

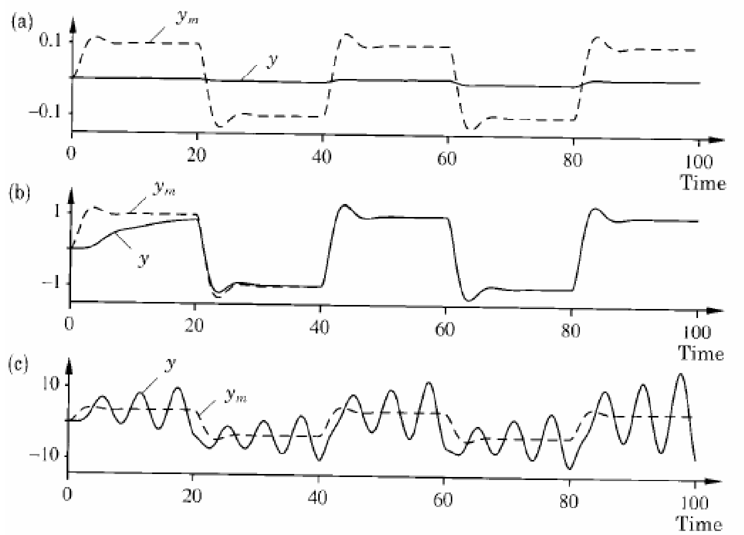


Figure 5.8 Simulation of the MRAS in Example 5.5. The command signal is a square wave with the amplitude (a) 0.1, (b) 1, and (c) 3.5. The model output y_m is a dashed line; the process output is a solid line. The following parameters are used: $k = a_1 = a_2 = \theta^0 = 1$, and $\gamma = 0.1$.

- رفع اشکال: الگوریتم های نرمالیزه شده

MIT Rule: $$\frac{d\theta}{dt} = \gamma \phi e, \quad \phi = -\frac{\partial e}{\partial \theta}$$

Normalized MIT Rule: $$\frac{d\theta}{dt} = \frac{\gamma \phi e}{\alpha + \phi^T \phi}, \quad \alpha > 0$$

معادله مشخصه سیستم حلقه بسته با قانون جدید:

$$s + \gamma \frac{\phi^o u_c^o}{\alpha + {\phi^o}^T \phi^o} kG(s) = 0$$

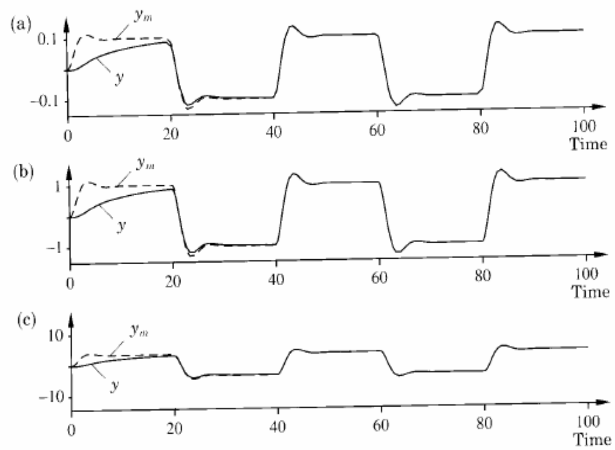


Figure 5.9 Simulation of the MRAS in Example 5.5 with the normalized MIT rule. The command signal is a square wave with the amplitude (a) 0.1, (b) 1, and (c) 3.5. Compare with Fig. 5.8. The model output y_m is a dashed line; the process output is a solid line. The parameters used are $k = a_1 = a_2 = \theta^0 = 1$, $\alpha = 0.001$, and $\gamma = 0.1$.

❖ سیستم های تطبیقی مدل مرجع (MRAS) بر اساس نظریه پایداری لیاپانوف

- تفاوت اصلی با روش قبل: مکانیزم تطابق
- روش های نسل دوم سیستم های تطبیقی: Stable MRAS
- روش دوم لیاپانوف
- طراحی MRAS بر اساس نظریه پایداری لیاپانوف
  - طراحی بر اساس فضای حالت
  - در دسترس بودن متغیرهای حالت

Lyapunov's Theory for Time-invariant Systems

Fundamental contributions to the stability theory for nonlinear systems were made by the Russian mathematician Lyapunov in the end of the nineteenth century. Lyapunov investigated the nonlinear differential equation

$$\frac{dx}{dt} = f(x) \qquad f(0) = 0 \tag{5.17}$$

Since $f(0) = 0$, the equation has the solution $x(t) = 0$. To guarantee that a solution exists and is unique, it is necessary to make some assumptions about $f(x)$. A sufficient assumption is that $f(x)$ is locally Lipschitz, that is,

$$\|f(x) - f(y)\| \leq L\|x - y\| \qquad L > 0$$

in the neighborhood of the origin. Lyapunov was interested in investigating whether the solution of Eq. (5.17) is stable with respect to perturbations. For this purpose he introduced the following stability concept.

DEFINITION 5.1 Lyapunov stability

The solution $x(t) = 0$ to the differential equation (5.17) is called *stable* if for given $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that all solutions with initial conditions

$$\|x(0)\| < \delta$$

have the property

$$\|x(t)\| < \varepsilon \quad \text{for} \quad 0 \leq t < \infty \tag{5.18}$$

The solution is *unstable* if it is not stable. The solution is *asymptotically stable* if it is stable and δ can be found such that all solutions with $\|x(0)\| < \delta$ have the property that $\|x(t)\| \to 0$ as $t \to \infty$. □

Remark 1. If the solution is asymptotically stable for any initial value, then it is said to be *globally asymptotically stable*.

Remark 2. Notice that Lyapunov stability refers to stability of a particular solution and not to the differential equation. □

DEFINITION 5.2 Positive definite and semidefinite functions

A continuously differentiable function $V : R^n \to R$ is called *positive definite* in a region $U \subset R^n$ containing the origin if

1. $V(0) = 0$
2. $V(x) > 0, \quad x \in U$ and $x \neq 0$

A function is called *positive semidefinite* if Condition 2 is replaced by $V(x) \geq 0$. □

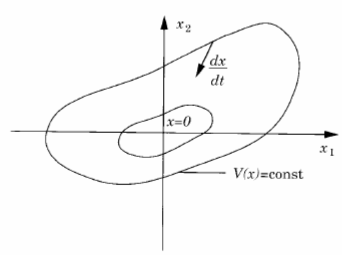


Figure 5.10 Illustration of Lyapunov's method for investigating stability.

THEOREM 5.1 Lyapunov's stability theorem: time-invariant systems
If there exists a function $V : R^n \to R$ that is positive definite such that its derivative along the solution of Eq. (5.17),

$$\frac{dV}{dt} = \frac{\partial V^T}{\partial x}\frac{dx}{dt} = \frac{\partial V^T}{\partial x} f(x) = -W(x) \tag{5.19}$$

is negative semidefinite, then the solution $x(t) = 0$ to Eq. (5.17) is stable. If dV/dt is negative definite, then the solution is also asymptotically stable. The function V is called a *Lyapunov function* for the system (5.17).

Moreover if

$$\frac{dV}{dt} < 0 \quad \text{and} \quad V(x) \to \infty \quad \text{when} \quad \|x\| \to \infty$$

then the solution is globally asymptotically stable.

Proof: Given $\varepsilon > 0$ such that $\{x \mid \|x\| \le \varepsilon\} \in U$, determine ℓ and δ such that

$$\ell = \min_{\|x\|=\varepsilon} V(x) = \max_{\|x\|\le\delta} V(x) \tag{5.20}$$

Consider initial conditions such that

$$\|x(0)\| < \delta$$

Since V is positive definite, it then follows from Definition 5.2 that

$$V(x(0)) < \ell$$

To prove that inequality (5.18) holds, we proceed by contradiction. Assume that t_1 is the smallest value such that $\|x(t_1)\| = \varepsilon$. It follows from Eq. (5.20) that

$$V(x(t_1)) \ge \ell$$

Furthermore,

$$V(x(t_1)) = V(x(0)) + \int_0^{t_1} \frac{dV}{dt}\,dt = V(x(0)) - \int_0^{t_1} W(x(s))\,ds \tag{5.21}$$

Since $W(x)$ is positive semidefinite, it follows that

$$V(x(t_1)) \le V(x(0)) < \ell$$

and we have thus obtained a contradiction and it can be concluded that $\|x(t)\| < \varepsilon$ for all t, which by Definition 5.1 implies that the solution $x(t) = 0$ is stable. To prove asymptotic stability, we notice that it follows from Eq. (5.21) that

$$0 \le \int_0^t W(x(s))\,ds = V(x(0)) - V(x(t)) \le \ell$$

Since $W(x)$ and $x(t)$ are continuous, it then follows that

$$\lim_{t\to\infty} W(x(t)) = 0$$

If $W(x)$ is positive definite, this implies that $x(t) \to 0$ as $t \to \infty$. $\square$

Remark. Notice that it follows from the proof that if the derivative of the Lyapunov function is negative semidefinite, the solution converges to the set $\{x \mid W(x) = 0\}$. $\square$

THEOREM 5.2 Lyapunov functions for linear systems
Assume that the linear system

$$\frac{dx}{dt} = Ax \tag{5.22}$$

is asymptotically stable. Then for each symmetric positive definite matrix Q there exists a unique symmetric positive definite matrix P such that

$$A^T P + PA = -Q \tag{5.23}$$

Furthermore, the function

$$V(x) = x^T P x \tag{5.24}$$

is a Lyapunov function for Eq. (5.22).

Proof: Let Q be a symmetric positive definite matrix. Define

$$P(t) = \int_0^t e^{A^T(t-s)} Q e^{A(t-s)}\, ds$$

The matrix P is symmetric and positive definite because an integral of positive definite matrices is positive definite. The matrix P also satisfies

$$\frac{dP}{dt} = A^T P + PA + Q$$

Since the matrix A is stable, the limit

$$P_o = \lim_{t\to\infty} P(t)$$

exists. This matrix satisfies Eq. (5.23). It can also be shown that the solution to Eq. (5.23) is unique, which completes the argument. □

Lyapunov Theory for Time-variable Systems

We now consider time-variable differential equations of the type

$$\frac{dx}{dt} = f(x,t) \tag{5.25}$$

The origin is an equilibrium point for Eq. (5.25) if $f(0,t) = 0\ \forall t \geq 0$. It is assumed that f is such that solutions exist for all $t \geq t_0$. To guarantee this, it is assumed that f is piecewise continuous in t and locally Lipschitz in x in a neighborhood of $x(t) = 0$. We now investigate the stability of the solution $x(t) = 0$.

In the time-varying case the solution will depend on t as well as on the starting time t_0. This implies that the bound δ in Definition 5.1 will depend on ε and t_0. The definition on stability can be refined to give uniform stability properties with respect to the initial time. We have the following definition.

DEFINITION 5.3 Uniform Lyapunov stability
The solution $x(t) = 0$ of Eq. (5.25) is *uniformly stable* if for $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$, independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \qquad \forall t \geq t_0 \geq 0$$

The solution is *uniformly asymptotically stable* if it is uniformly stable and there is $c > 0$, independent of t_0, such that $x(t) \to 0$ as $t \to \infty$, uniformly in t_0, for all $\|x(t_0)\| < c$. □

DEFINITION 5.4 Class K function

A continuous function α: $[0, a) \to [0, \infty)$ is said to belong to *class K* if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. □

For time-varying systems the following stability theorem can now be stated.

THEOREM 5.3 Lyapunov's stability theorem: Time-varying systems

Let $x = 0$ be an equilibrium point for Eq. (5.25) and $D = \{x \in R^n \mid \|x\| < r\}$. Let V be a continuously differentiable function such that

$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|) \tag{5.26}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \le -\alpha_3(\|x\|)$$

for $\forall t \ge 0$, where α_1, α_2, and α_3 are class K functions. Then $x = 0$ is uniformly asymptotically stable.

Proof: A proof can be found in Khalil (1992). □

Remark 1. The derivative of V along the trajectories of Eq. (5.25) is now given by

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t)$$

Remark 2. A function $V(x,t)$ satisfying the left inequality of (5.26) is said to be positive definite. A function satisfying the right inequality of (5.26) is said to be *decrescent*.

Remark 3. To show stability for time-variable systems, it is necessary to bound the function $V(x,t)$ by a function that doesn't depend on t. □

When using Lyapunov theory on adaptive control problems, we often find that dV/dt only is negative semidefinite. This implies that additional conditions must be imposed on the system. The following lemma gives a useful result.

LEMMA 5.1 Barbalat's lemma

If g is a real function of a real variable t, defined and uniformly continuous for $t \ge 0$, and if the limit of the integral

$$\int_0^t g(s)\, ds$$

as t tends to infinity exists and is a finite number, then

$$\lim_{t \to \infty} g(t) = 0$$ □

Remark. A consequence of Barbalat's lemma is that if $g \in L_2$ and dg/dt is bounded, then

$$\lim_{t \to \infty} g(t) = 0$$ □

When applying Lyapunov theory to an adaptive control problem, we get a time derivative of the Lyapunov function V, which depends on the control signal and other signals in the system. If these signals are bounded, Lemma 5.1 and the remark that follows can be used on dV/dt to prove stability. We have the following theorem.

THEOREM 5.4 Boundedness and convergence set

Let $D = \{x \in R^n \mid \|x\| < r\}$ and suppose that $f(x,t)$ is locally Lipschitz on $D \times [0, \infty)$. Let V be a continuously differentiable function such that

$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|)$$

and

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -W(x) \leq 0$$

$\forall t \geq 0$, $\forall x \in D$, where α_1 and α_2 are class K functions defined on $[0, r)$ and $W(x)$ is continuous on D. Further, it is assumed that dV/dt is uniformly continuous in t.

Then all solutions to Eq. (5.25) with $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$ are bounded and satisfy

$$W(x(t)) \to 0 \quad \text{as} \quad t \to \infty$$

Moreover, if all the assumptions hold globally and α_1 belongs to class K_∞, the statement is true for all $x(t_0) \in R^n$. □

- یک مثال و ایده کلی

System: $\frac{dy}{dt} = -ay + bu$

Model Reference: $\frac{dy_m}{dt} = -a_m y_m + b_m u_c$

Controller: $u(t) = t_0 u_c(t) - s_0 y(t)$

Error: $e(t) = y(t) - y_m(t)$

Error Dynamics:

$$\begin{aligned}\dot{e} = \dot{y} - \dot{y}_m &= -ay + bu + a_m y_m - b_m u_c \\ &= -ay + bt_0 u_c(t) - bs_0 y(t) - a_m e + a_m y - b_m u_c \\ &= -a_m e + y(t)(a_m - bs_0 - a) + u_c(t)(bt_0 - b_m)\end{aligned}$$

اگر

$$t_0 = \frac{b_m}{b}, s_0 = \frac{a_m - a}{b}$$

مقادیر مطلوب خود باشند آنگاه

$$e(t) \to 0$$

و ردیابی انجام خواهد شد.

✸مکانیزم تطابقی برای پارامترها طراحی کنید که آنها را به سمت مقادیر مطلوب میل دهد.

✓ تعریف یک تابع لیاپانف:

$$V(e, t_0, s_0) = \frac{1}{2}[e^2 + \frac{1}{b\gamma}(bs_0 + a - a_m)^2 + \frac{1}{b\gamma}(bt_0 - b_m)^2]$$

به نقطه تعادل توجه کنید. داریم:

$$\frac{dV}{dt} = e\frac{de}{dt} + \frac{1}{\gamma}(bs_0 + a - a_m)\frac{ds_0}{dt} + \frac{1}{\gamma}(bt_0 - b_m)\frac{dt_0}{dt}$$

$$= -a_m e^2 + \frac{1}{\gamma}(bs_0 + a - a_m)[\frac{ds_0}{dt} - \gamma ye] + \frac{1}{\gamma}(bt_0 - b_m)[\frac{dt_0}{dt} + \gamma u_c e]$$

به روز کردن پارامترها به صورت:

$$\frac{dt_0}{dt} = -\gamma u_c e, \quad \frac{ds_0}{dt} = \gamma ye$$

می دهد:

$$\boxed{\frac{dV}{dt} = -a_m e^2}$$

✓ $\frac{dV}{dt}$ یک تابع منفی نیمه معین است.

✓ برای $e \neq 0$ یک تابع کاهشی است.

✓ داریم:

$$\boxed{\frac{d^2V}{dt^2} = -2a_m e \frac{de}{dt}}$$

✓ از آنجاییکه e و $\frac{de}{dt}$ کران دارند لذا

$$\frac{d^2V}{dt^2} < \infty$$

ازاینرو $e \to 0$ و سیستم پایدار است.

✓ همگرایی پارامترها ؟

✓ قانون تطابق:

بردار پارامترها

$$\frac{d\theta}{dt} = \gamma\phi e$$

$$\boxed{= \frac{1}{p+a_m}\begin{bmatrix}-u_c \\ y\end{bmatrix} \to \text{MIT}}$$

$$\boxed{= \begin{bmatrix}-u_c \\ y\end{bmatrix} \to \text{Lyapunov}}$$

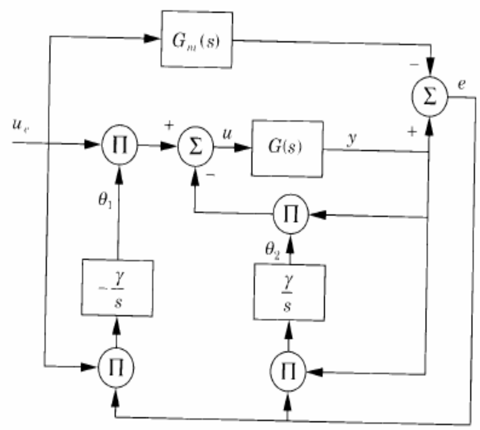


Figure 5.11 Block diagram of an MRAS based on Lyapunov theory for a first-order system. Compare with the controller based on the MIT rule for the same system in Fig. 5.4.

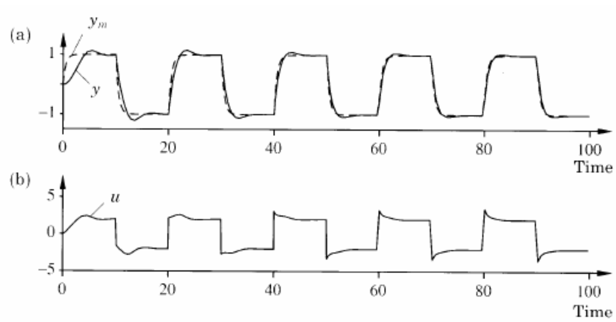


Figure 5.12 Simulation of the system in Example 5.7 using an adaptive controller based on Lyapunov theory. The parameter values are $a = 1$, $b = 0.5$, $a_m = b_m = 2$, and $\gamma = 1$. (a) Process (solid line) and model (dashed line) outputs. (b) Control signal.

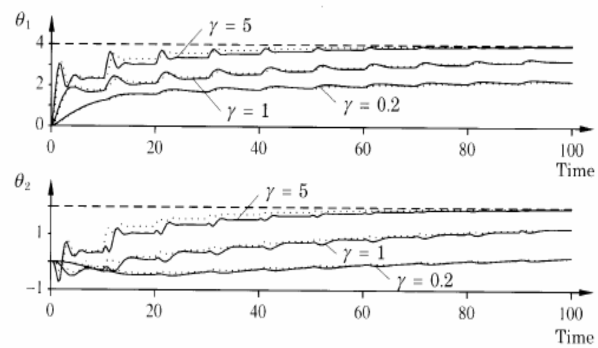


Figure 5.13 Controller parameters θ_1 and θ_2 for the system in Example 5.7 when $\gamma = 0.2$, 1, and 5. The dotted lines are the parameters obtained with the MIT rule. Compare Fig. 5.6.

- مثال دیگر: MRAS با روش لیاپانوف

فرایند: $(b_0, a_1, a_2$ Unknown$)$

$$g(s) = \frac{b_0}{s^2 + a_1 s + a_2}$$

مدل مرجع:

$$g_m(s) = \frac{b_m}{s^2 + a_{1m} s + a_{2m}}$$

کنترل:

$$u = f u_c - q_0 \dot{y} - q_1 y$$

سیستم حلقه بسته:

$$y = \frac{f b_0}{s^2 + (a_1 + b_0 q_0)s + (a_2 + b_0 q_1)} u_c$$

پارامترهای کنترل:

$$f, q_0, q_1$$

خطای سیستم تطبیقی:

$$e = y - y_m$$

خطای پارامترها:

$$\tilde{b}_0 = f b_0 - b_m$$
$$\tilde{a}_1 = a_1 + b_0 q_0 - a_{m1}$$
$$\tilde{a}_2 = a_2 + b_0 q_1 - a_{m2}$$

داریم:

$$\ddot{y} + (a_1 + b_0 q_0)\dot{y} + (a_2 + b_0 q_1) y = f b_0 u_c$$
$$\ddot{y}_m + \quad a_{m1}\dot{y}_m + \quad a_{m2} y_m = b_0 u_c$$

لذا برای دینامیک خطا به دست می آوریم:

$$\ddot{e} + a_{m1}\dot{e} + a_{m2} e = \tilde{b}_0 u_c - \tilde{a}_1 \dot{y} - \tilde{a}_2 y$$

تابع لیاپانوف:

$$V = a_{m2} e^2 + \dot{e}^2 + \frac{1}{\gamma_0}\tilde{b}_0^2 + \frac{1}{\gamma_1}\tilde{a}_1^2 + \frac{1}{\gamma_2}\tilde{a}_2^2 \qquad (\gamma_0, \gamma_1, \gamma_2 \in \Re^+)$$

لذا:

$$\dot{V} = 2a_{m2} e\dot{e} + 2\dot{e}(-a_{m1}\dot{e} - a_{m2} e + \tilde{b}_0 u_c - \tilde{a}_1 \dot{y} - \tilde{a}_2 y) + \frac{2}{\gamma_0}\tilde{b}_0 \dot{\tilde{b}}_0 + \frac{2}{\gamma_1}\tilde{a}_1 \dot{\tilde{a}}_1 + \frac{2}{\gamma_2}\tilde{a}_2 \dot{\tilde{a}}_2$$
$$= -2a_{m1}\dot{e}^2 + 2\tilde{b}_0[\dot{e} u_c + \frac{1}{\gamma_0}\dot{\tilde{b}}_0] + 2\tilde{a}_1[-\dot{e}\dot{y} + \frac{1}{\gamma_1}\dot{\tilde{a}}_1] + 2\tilde{a}_2[-\dot{e} y + \frac{1}{\gamma_2}\dot{\tilde{a}}_2]$$

برای آن که مشتق تابع لیاپانوف منفی نیمه معین باشد باید داشته باشیم:

$$\dot{\tilde{b}}_0 = -\gamma_0 \dot{e} u_c \Rightarrow f = -\frac{\gamma_0}{b_0}\int_0^{t_0} \dot{e} u_c dt + f(0)$$

$$\dot{\tilde{a}}_1 = \gamma_1 \dot{e} \dot{y} \quad \Rightarrow q_0 = \frac{\gamma_1}{b_0}\int_0^{t_0} \dot{e} \dot{y} dt + q_0(0)$$

$$\dot{\tilde{a}}_2 = \gamma_1 \dot{e} y \quad \Rightarrow q_1 = \frac{\gamma_2}{b_0}\int_0^{t_0} \dot{e} y dt + q_1(0)$$

- ✓ معلوم بودن علامت ضریب.
- ✓ تنها Global Stability و نه Global Asymptotic Stability تضمین شده است.
- ✓ شرایط همگرایی پارامترها.

- **سیستم های فضای حالت**

- ✓ MRAS پایدار برای سیستم های خطی کلی:
  - تعیین ساختار کنترل
  - تعیین معادله خطا
  - تعیین تابع لیاپانوف و کاربرد آن درتعیین قانون تطبیق پارامترها

- ✓ تحلیل و طراحی در حوزه فضای حالت
- ✓ رویتگرهای تطبیقی

سیستم:

$$\dot{x} = Ax + Bu$$

مدل مرجع:

$$\dot{x}_m = A_m x_m + B_m u_c$$

کنترل:

$$u = Mu_c - Lx$$

حلقه بسته:

$$\dot{x} = (A - BL)x + BMu_c = \underbrace{A_c(\theta)}_{\boxed{A_m \text{ at } \theta = \theta^0}} x + \underbrace{B_c(\theta)}_{\boxed{B_m \text{ at } \theta = \theta^0}} u_c$$

- معادله خطا

$$e = x - x_m$$

$$\Rightarrow \dot{e} = \dot{x} - \dot{x}_m = Ax + Bu - A_m x_m - B_m u_c$$

$$\pm A_m x \Rightarrow$$

$$\dot{e} = A_m e + (A - A_m - BL)x + (BM - B_m)u_c$$

$$= A_m e + (A_c(\theta) - A_m)x + (B_c(\theta) - B_m)u_c$$

$$= A_m e + \Psi(\theta - \theta^0)$$

تابع لیاپانوف

$$V(e,\theta) = \frac{1}{2}(\gamma e^T \underbrace{P}_{\text{ماتریس PD}} e + (\theta - \theta^0)^T(\theta - \theta^0))$$

لذا:

$$\frac{dV(e,\theta)}{dt} = -\frac{1}{2}\gamma e^T Qe + \gamma(\theta-\theta^0)^T \Psi^T Pe + (\theta-\theta^0)^T \frac{d\theta}{dt}$$
$$= -\frac{1}{2}\gamma e^T Qe + (\theta-\theta^0)^T \left(\frac{d\theta}{dt} + \gamma\Psi^T Pe\right)$$

که در آن

Q is PD and $\quad A_m^T P + PA_m = -Q$

با انتخاب

$$\frac{d\theta}{dt} = -\gamma\Psi^T Pe$$

داریم

$$\frac{dV(e,\theta)}{dt} = -\frac{1}{2}\gamma e^T Qe$$

- مثال (تطابق بهره پیشرو)

$e = G(p)(\theta-\theta^0)u_c$

Let, $\quad (A,B,C) \leftrightarrow G(s)$

$\Rightarrow \dot{x} = Ax + B(\theta-\theta^0)u_c$

$\quad e = Cx$

If $\dot{x} = Ax$ is stable, $\exists P,Q$ s.t. $\quad A^T P + PA = -Q$

Lyapunov Function:

$$V = \frac{1}{2}(\gamma x^T Px + (\theta-\theta^o)^2)$$

$$\Rightarrow \frac{dV}{dt} = \frac{\gamma}{2}(\dot{x}^T Px + x^T P\dot{x}) + (\theta-\theta^o)\frac{d\theta}{dt} = -\frac{\gamma}{2}x^T Qx + (\theta-\theta^o)\left(\frac{d\theta}{dt} + \gamma u_c B^T Px\right)$$

$$\Rightarrow \qquad \frac{d\theta}{dt} = -\gamma u_c B^T Px \qquad (\theta \overset{?}{\rightarrow} \theta^o)$$

• فیدبک خروجی

اگر داشته باشیم:

$$B^T P = C$$
$$\Rightarrow B^T Px = Cx = e$$
$$\Rightarrow \frac{d\theta}{dt} = -\gamma u_c e$$

✓ چه شرایطی برای وجود ماتریس P بایستی برقرار باشد؟

✓ ایده SPR (Strictly Positive Real) بودن سیستم

• تعریف توابع تبدیل حقیقی مثبت Positive Real (PR) TFN

تابع تبدیل گویای $G(s)$ با ضرایب حقیقی را حقیقی مثبت گویند اگر:

$$\boxed{\operatorname{Re} G(s) \geq 0 \qquad \text{for} \quad s \geq 0}$$

و آن را اکیدا حقیقی مثبت می نامند (SPR) اگر $G(s-\varepsilon)$ برای یک $\varepsilon \geq 0$ حقیقی مثبت باشد.

مثال

$$G(s) = \frac{1}{s+1} \quad \text{is SPR}$$
$$G(s) = \frac{1}{s} \quad \text{is PR but not SPR}$$

- لم KALMAN-YAKUBOVICH

سیستم LTI می نیمال زیر را در نظر بگیرید

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

تابع تبدیل متناظر سیستم $G(s)$ SPR است اگر و فقط اگر

$$A^T P + PA = -Q$$
$$and$$
$$B^T P = C$$

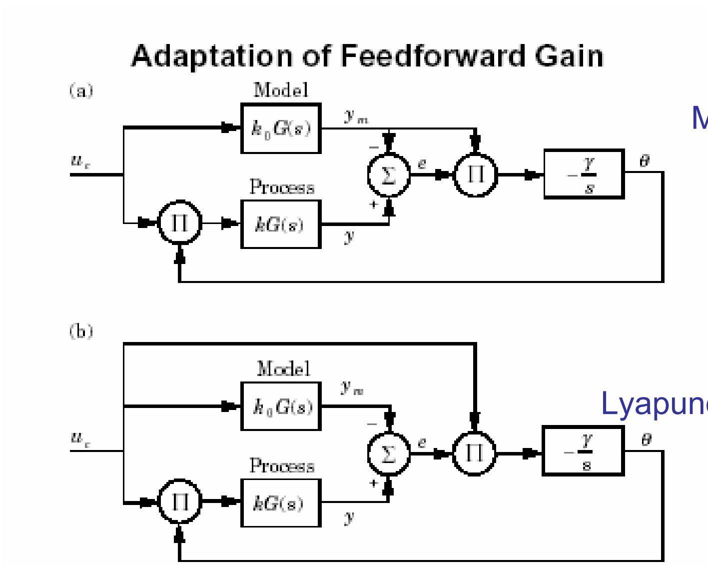
- قضیه MRAS با استفاده از قاعده لیاپانوف

در مساله تطابق بهره با فرض اکیدا حقیقی مثبت بودن سیستم قاعده تطابق پارامتر زیر

$$\frac{d\theta}{dt} = -\gamma u_c e \qquad \gamma > 0$$

خطای خروجی در معادله زیر به صفر میل می کند:

$$\dot{x} = Ax + B(\theta - \theta^0)u_c$$
$$e = Cx$$



- قاعده MIT
- قاعده Lyapunov

- **تنظیم PI**

✓ قواعد تنظیم از نوع رگلاتورهای انتگرال

✓ پارامترها خروجی انتگرل گیر

$$\frac{d\theta}{dt} = -\gamma u_c e$$

✓ قوانین تنظیم تناسبی + انتگرالی

$$\theta(t) = -\gamma_1 u_c(t)e(t) - \gamma_2 \int_0^t u_c(\tau)e(\tau)d\tau$$

$$\Rightarrow H(s) = \gamma_1 + \gamma_2 \frac{1}{s} \qquad \text{a PR TFN}$$

✓ تطابق سریعتر

❖ سیستم های تطبیقی مدل مرجع (MRAS) بر اساس نظریه پایداری BIBO

- دو نگاه سیستمی:
  - *دیدگاه داخلی یا فضای حالت*
  - *دیدگاه خارجی یا جعبه سیاه ورودی-خروجی*

- دو روش کلی تحلیل پایداری:
  - *روش لیاپانوف*
  - *روش ورودی خروجی*

- دیدگاه اپراتوری سیستم های دینامیکی

$$S : X \to X$$

سیستم

فضای نرم دار

انتخاب نرم؟

$$\|u\| = \sqrt{\int_{-\infty}^{\infty} u^2(t)dt}$$

سیگنال های مورد استفاده در کنترل؟

$$or$$

$$\|u\| = \sup_{0 \le t \le \infty} |u(t)|$$

- سیگنال برش داده شده

$$x_T(t) = \begin{cases} x(t) & 0 \le t \le T \\ 0 & t > T \end{cases}$$

- فضای بسط داده شده

If X is a normed linear subspace of Y, then the extended space X_e is the set $\{x \in Y \mid x_T \in X \text{ for some fixed } T > 0\}$

- بهره سیستم غیر خطی

$$\gamma(S) = \sup_{u \in x_e} \frac{\|Su\|}{\|u\|}$$

بهره سیستم غیر خطی کوچکترین مقداری است که:

$$\|Su\| \le \gamma(S)\|u\| \qquad \text{for all } u \in x_e$$

✓ چند مثال

EXAMPLE 5.8 Linear systems with signals in L_{2e}

Let the signal space be L_{2e}. Consider a linear system with the transfer function $G(s)$. Assume that $G(s)$ has no poles in the closed right half-plane and that the system is initially at rest. Let u be the input and y the output, and let U and Y be the corresponding Laplace transforms. It follows from Parseval's theorem, Theorem 2.8, that

$$\begin{aligned}\|y\|^2 &= \int_0^\infty y^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(i\omega)Y(-i\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega)U(i\omega)G(-i\omega)U(-i\omega)\,d\omega \\ &\le \max_\omega |G(i\omega)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} U(i\omega)U(-i\omega)\,d\omega \\ &= \max_\omega |G(i\omega)|^2 \int_0^\infty u^2(t)\,dt = \max_\omega |G(i\omega)|^2 \cdot \|u\|^2\end{aligned}$$

Hence

$$\|y\| \le \max_\omega |G(i\omega)| \cdot \|u\|$$

The gain is thus less than $\max |G(i\omega)|$. We get equality in the above equation if u is a sinusoid with the frequency that maximizes $|G(i\omega)|$. However, such a signal is not in L_{2e}. The value of $\|y\|$ can be made arbitrarily close to $\max |G(i\omega)|$ with a truncated sinusoid in L_{2e} by making T sufficiently large. The gain of the system is thus

$$\gamma(G) = \max_\omega |G(i\omega)| \tag{5.41}$$

□

EXAMPLE 5.9 Linear system with sup norm

Consider a stable linear system with impulse response $h(t)$. We have

$$y(t) = \int_0^\infty h(\tau)u(t-\tau)\,d\tau$$

Using the sup norm, we get

$$|y(t)| = \left|\int_0^\infty h(\tau)u(t-\tau)\,d\tau\right| \le \sup_t |u(t)| \int_0^\infty |h(\tau)|\,d\tau$$

This gives

$$\sup_t |y(t)| \le \gamma(G) \cdot \sup_t |u(t)|$$

where the gain of the system is given by

$$\gamma(G) = \int_0^\infty |h(\tau)|\,d\tau$$

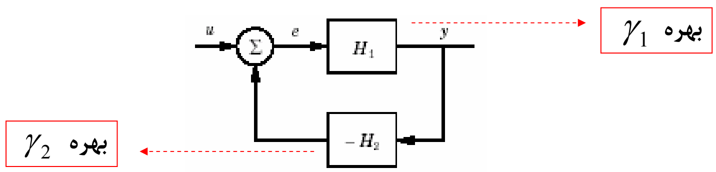
If we let $u_0 = \max_t |u(t)|$, the maximum is assumed for the signal

$$u(s) = u_0 \operatorname{sign}(h(t-s))$$

However, this signal is not in L_{2e}. Since the system is stable, we can get arbitrarily close with a signal in L_{2e} by making T sufficiently large. □

❖ قضیه بهره کوچک (The Small Gain Theorem)

- تعریف سیستم پایدار ورودی-خروجی (BIBO) است اگر بهره آن کراندار باشد.
- قضیه سیستم حلقه بسته زیر پایدار BIBO است



اگر

$$\gamma_1\gamma_2 < 1$$

و بهره آن کمتر از مقدار زیر است

$$\gamma = \frac{\gamma_1}{1-\gamma_1\gamma_2}$$

❖ پسیو بودن (Passivity)

- ایده پایداری دیگر بر اساس دیدگاه ورودی- خروجی
- پسیو بودن فرموله سازی مجردی از اتلاف انرژی (Energy Dissipation)
- سیستم های الکتریکی
- سیستم های مکانیکی
- سیستم های خطی و غیر خطی
- ارایه یک معیار کلی پایداری
- Passivity $\longleftrightarrow$ Lyapunov

- ضرب داخلی

$$\langle x | y \rangle = \int_0^{\infty} x(s) y(s) ds = \int_0^{T} x(s) y(s) ds$$

- ایده فاز برای ورودی داده شده:

$$\cos\varphi = \frac{\langle y | u \rangle}{\|u\| \|y\|} = \frac{\langle Hu | u \rangle}{\|u\| \|Hu\|}$$

- تعریف

A system with input u and output y is *passive* if

$$\langle y | u \rangle \geq 0$$

The system is *input strictly passive* (ISP) if there exists $\varepsilon > 0$ such that

$$\langle y | u \rangle \geq \varepsilon \|u\|^2$$

and *output strictly passive* (OSP) if there exists $\varepsilon > 0$ such that

$$\langle y | u \rangle \geq \varepsilon \|y\|^2$$

یا سیستم Passive است اگر

$$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

❖ توابع حقیقی مثبت

- ایده پسیو بودن در سیستم های خطی مرتبط با PR or SPR TFNs
- تابع تبدیل سیستم خطی Passive or Dissipative or PR است اگر

$$-\frac{\pi}{2} \le \arg G(s) \le \frac{\pi}{2}, \quad \text{for all } \operatorname{Re} s \ge 0$$

- تابع تبدیل SPR
- نکته: سیستم های پسیو را می توان با جبرانسازهای SPR پایدار مقاوم کرد. زیرا

$$-\pi \le \arg C(s)G(s) \le \pi, \quad \text{for all } \operatorname{Re} s \ge 0$$

- مثال سیستم جرم و فنر با ورودی <u>نیرو</u> و خروجی <u>سرعت</u> :

$$M\ddot{x} = -kx + F$$

Let, $u = F, y = \dot{x}$

Then, Mechanical Power$=uy$

And, $G(s) = \dfrac{V(s)}{F(s)} = \dfrac{s}{Ms^2 + k}$ is PR TFN

- علاوه بر لم کالمن - یاکوبویچ دیگر شرایط (اکیدا) حقیقی مثبت بودن؟

Characterizing Positive Real Transfer Functions

THEOREM 2
A rational transfer function $G(s)$ with real coefficients is PR if and only if the following conditions hold.

- (i) The function has no poles in the right half-plane.
- (ii) If the function has poles on the imaginary axis or at infinity, they are simple poles with positive residues.
- (iii) The real part of G is nonnegative along the $i\omega$ axis, that is,

$$\mathrm{Re}\,(G(i\omega)) \geq 0$$

A transfer function is SPR if conditions (i) and (iii) hold and if condition (ii) is replaced by the condition that $G(s)$ has no poles or zeros on the imaginary axis. □

Examples in Passivity

$$\langle y \,|\, u \rangle = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\,\{G(i\omega)\}\, U(i\omega) U(-i\omega)\, d\omega$$

- Positive real (PR) if Re $G(i\omega) \geq 0$;
- Input strictly passive (ISP) if Re $G(i\omega) \geq \varepsilon > 0$;
- Output stricly passive (OSP) if Re $G(i\omega) \geq \varepsilon |G(i\omega)|^2$;

$G(s) = s + 1$ SPR and ISP not OSP

$G(s) = \frac{1}{s+1}$ SPR and OSP not ISP

$G(s) = \frac{s^2+1}{(s+1)^2}$ OSP and ISP not OSP

$G(s) = \frac{1}{s}$ PR not SPR, OPS or ISP

Nonlinear Static Systems $y = f(u)$

$$\langle y | u \rangle = \int_0^\infty f(u(t))u(t)\, dt$$

- Passive if $xf(x) \geq 0$
- Input strictly passive (ISP) if $xf(x) \geq \delta |x|^2$
- Output strictly passive if

$$xf(x) \geq \delta f^2(x)$$

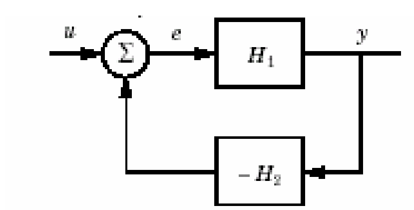
Geometric Interpretation

Example

- $f(x) = x + x^3$ input strictly passive
- $f(x) = x/(1+|x|)$ output strictly passive.

- قضیه Passivity

سیستم زیر



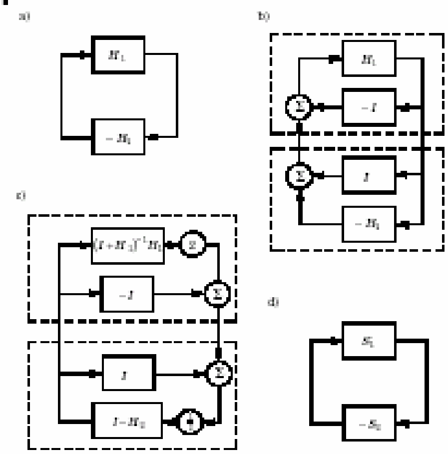
که در آن

H_1 Output Strictly Passive

H_2 Passive

پایدار BIBO است.

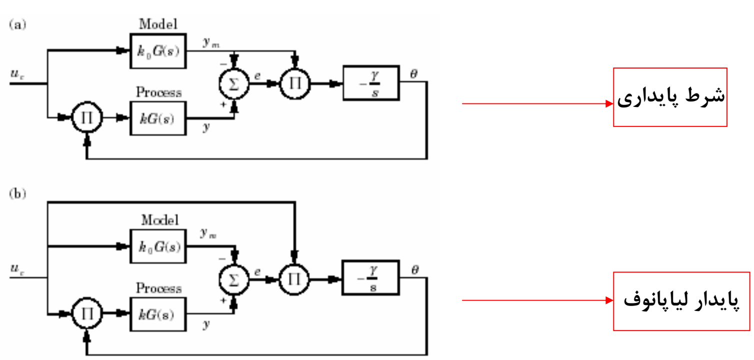
Relationships: Small Gain vs. Passivity Theorems



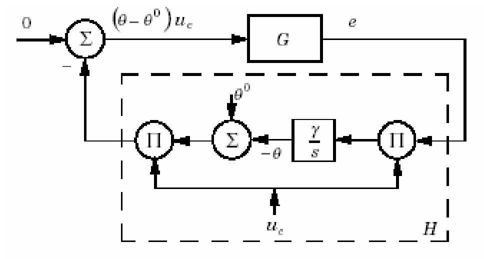
$$a \to b : H_1 \to (I + H_1)^{-1} H_1, H_2 \to I - H_2; \quad b \to c : \quad S_i = (H_i + I)^{-1}(H_i - I)$$

❖ کاربرد در سیستم های تطبیقی پایدار

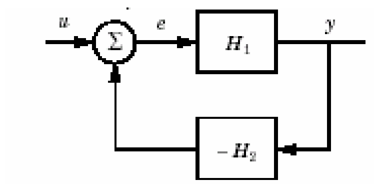
- **مثال** تطابق بهره پیشرو



رسم مجدد



مقایسه کنید با:



- لم اگر r تابع انتگرال پذیر مربعی و $G(s)$ مثبت حقیقی آنگاه سیستم داده شده با

$$y = r(G(p)ru)$$

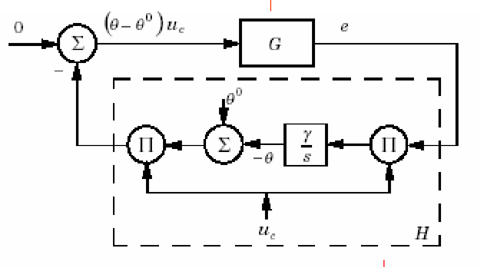
پسیو است.

اثبات

$$\langle y | u \rangle = \int_0^\infty y(\tau)u(\tau)d\tau = \int_0^\infty [u(\tau)r(\tau)][G(p)ru](\tau)d\tau$$
$$= \int_0^\infty w(\tau)[G(p)w](\tau)d\tau = \langle w | Gw \rangle$$

- نگاه مجدد:

تابع تبدیل سیستم SPR+Proper
و ورودی مرجع کراندار است و لذا OSP است



تابع تبدیل انتگرالی PR است و با توجه
به لم قبل سیستم غیر خطی پسیو است

لذا سیستم حلقه بسته *پایدار* است.

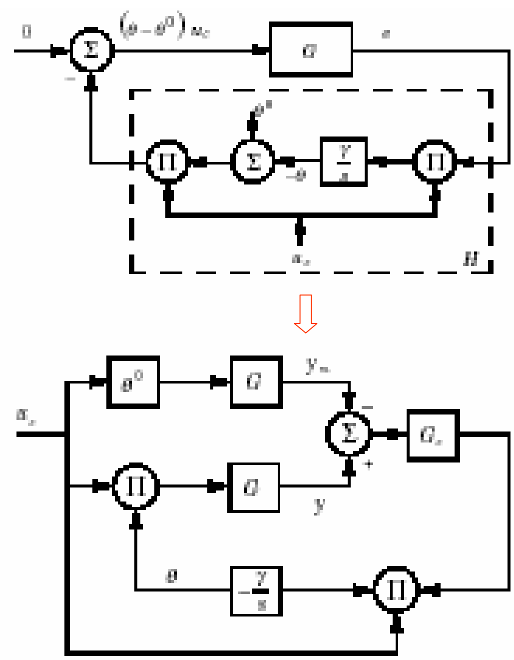
- نتیجه مهم: قضیه پسیویتی راه آسانی برای تولید قواعد تنظیم پایدار فراهم می کند. اگر جبرانسازی سری با سیستم معرفی شود که

$$e(t) \quad \text{to} \quad (\theta-\theta^0)u_c(t) \quad \text{Positive Real}$$

سیستم حلقه بسته پایدار خواهد بود. توجه کنید که:

$$G_c(s)G(s)=\frac{c(s)}{b(s)}\frac{b(s)}{a(s)}$$

شمای کلی
جبران سازی



- یک مشکل: درجه نسبی بزرگتر از یک !
- راه حل: خطای افزوده شده (Error Augmentation)

Factor

$$G = G_1 G_2$$

where the transfer function G_1 is SPR. The error $e = y - y_m$ can then be written as

$$\begin{aligned} e &= G(\theta - \theta^0)u_c = (G_1 G_2)(\theta - \theta^0)u_c \\ &= G_1\left(G_2(\theta - \theta^0)u_c + (\theta - \theta^0)G_2 u_c - (\theta - \theta^0)G_2 u_c\right) \end{aligned}$$

Introduce

$$\varepsilon = e + \eta$$

where η is the *error augmentation* defined by

$$\eta = G_1(\theta - \theta^0)G_2 u_c - G(\theta - \theta^0)u_c = G_1(\theta G_2 u_c) - G\theta u_c$$

Use adaptation law

$$\frac{d\theta}{dt} = -\gamma \varepsilon G_2 u_c$$

- قضیه (پایداری با استفاده از خطای افزوده شده)

سیستم تطبیقی مدل مرجع برای تطبیق بهره با:

$G = G_1 G_2 \quad (G_1$ SPR and BIBO Inverse Stable)

Then,

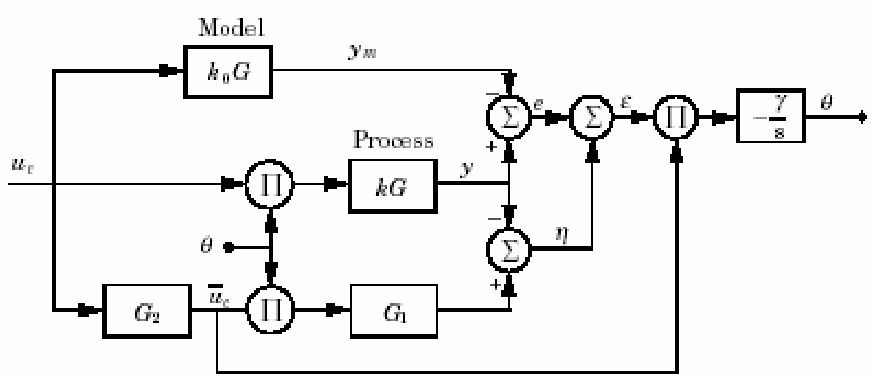
$$\frac{d\theta}{dt} = -\gamma\varepsilon(G_2 u_c)$$

Where,

$$\varepsilon = e + G_1(\theta G_2 u_c) - G(\theta u_c)$$

سیستم حلقه بسته ای می دهد که $\lim_{t\to\infty} e(t) \to 0$ و اگر u_c PE باشد و
تابع G_2 می نیمم فاز باشد آنگاه $\lim_{t\to\infty}(\theta - \theta^0) \to 0$.

MRAS with Augmented Error



❖ MRAS پایدار با فیدبک خروجی بر اساس نظریه Passivity

- طراحی برای سیستم های خطی کلی تک ورودی تک خروجی
- سه مرحله کلی طراحی:

مرحله ۱ ساختار کنترلی پیدا کنید که دنبال روندگی ایده ال مدل را برای سیستم تضمین کند.

مرحله ۲ تعیین مدل خطا

$$\varepsilon = G_1(p)[\phi^T(t)(\theta^0 - \theta)]$$

تابع تبدیل SPR — پارامترهای صحیح — پارامترهای واقعی

مرحله ۳ قانون تنظیم پارامتر با قانون SPR

$$\frac{d\theta}{dt} = \gamma\phi\varepsilon \quad \text{or} \quad \frac{d\theta}{dt} = \gamma\frac{\phi\varepsilon}{\alpha + \phi^T\phi}$$

- سیستم SISO-MP

$$Ay = Bu$$

and,

$$B = b_0 B^+$$

تکین و پایدار

بهره لحظه ای

- مدل مرجع

$$A_m y_m = B_m u_c$$

- ساختار رگلاتور

$$Ru = Tu_c - Sy$$

$$R = R_1 B^+ \quad T = A_o B_m / b_0$$

and

$$AR_1 + b_0 S = A_o A_m$$

- مدل خطا

$$A_oA_m y = AR_1 y + b_0 Sy = R_1 Bu + b_0 Sy = b_0(Ru + Sy)$$

معادله مدل می دهد

$$A_oA_m y_m = A_o B_m u_c = b_0 T u_c$$

لذا

$$A_oA_m e = A_oA_m (y - y_m) = b_0(Ru + Sy - Tu_c)$$

و

$$e = \frac{b_0}{\underline{A_oA_m}}(Ru + Sy - Tu_c)$$

$$\downarrow$$

$$\boxed{\text{تابع تبدیل SPR نیست}}$$

خطای فیلتر شده

$$e_f = \frac{Q}{P} e = \frac{Q}{P}(y - y_m)$$

$$\downarrow$$

$$\frac{b_0 Q}{\underline{A_oA_m}}$$

$$\downarrow$$

$$\boxed{\text{تابع تبدیل SPR است}}$$

لذا

$$e_f = \frac{b_0 Q}{A_oA_m}\left(\frac{R}{P}u + \frac{S}{P}y - \frac{T}{P}u_c\right)$$

با

$$P = P_1 P_2, \deg P_2 = \deg R \text{ and Monic}$$

داریم

$$\frac{R}{P} = \frac{R - P_2 + P_2}{P_1 P_2} = \frac{1}{P_1} + \frac{R - P_2}{P}$$

و لذا

$$e_f = \frac{b_0 Q}{A_o A_m}\left(\frac{1}{P_1}u + \frac{R - P_2}{P}u + \frac{S}{P}y - \frac{T}{P}u_c\right)$$

توجه کنید که

$$\deg T = m, \deg S = l, \deg R = k$$

- با تعریف

$$\theta^0 = \begin{bmatrix} r_1' & \cdots & r_k' & s_0 & \cdots & s_l & t_0 & \cdots & t_m \end{bmatrix}^T$$

$$\phi^T = \Bigg[\underbrace{\frac{p^{k-1}}{P(p)}u \quad \cdots \quad \frac{1}{P(p)}u}_{\text{ورودی فیلتر شده}} \quad \underbrace{\frac{p^l}{P(p)}y \quad \cdots \quad \frac{1}{P(p)}y}_{\text{خروجی فیلتر شده}} \quad \underbrace{-\frac{p^m}{P(p)}u_c \quad \cdots \quad -\frac{1}{P(p)}u_c}_{\text{ورودی مرجع فیلتر شده}} \Bigg]$$

داریم

$$e_f = \frac{b_0 Q}{A_o A_m}\left(\frac{1}{P_1}u + \phi^T \theta^0\right)$$

- قانون کنترل

$$u = -P_1(\phi^T \theta^0) = -P_1((\theta^0)^T \phi) = -(\theta^0)^T (P_1\phi)$$

چند جمله ای در اپراتور دیفرانسیلی

لذا قانون کنترل قابل تحقق

$$\boxed{u = -(\theta)^T (P_1\phi)}$$

با این کنترل داریم

$$e_f = \frac{b_0 Q}{A_o A_m}(\phi^T \theta^0 - \frac{1}{P_1}\theta^T (P_1\phi))$$

$$= \frac{b_0 Q}{A_o A_m}(\phi^T \theta^0 - \phi^T \theta - \frac{1}{P_1}\theta^T (P_1\phi) + \phi^T \theta)$$

با تعریف

$$\eta = \frac{1}{P_1}\theta^T (P_1\phi) - \phi^T \theta = -(\frac{1}{P_1}u + \phi^T \theta)$$

و

$$\varepsilon = e_f + \frac{b_0 Q}{A_o A_m}\eta = \frac{b_0 Q}{A_o A_m}\phi^T (\theta^0 - \theta)$$

خطای افزوده شده
Augmented Error

تابع تبدیل **SPR** است

محاسبه خطای افزوده شده:

$$\varepsilon = \frac{Q}{P}(y - y_m) + \frac{b_0 Q}{A_o A_m}\eta$$

• پایداری حلقه بسته:

$$\frac{b_0 Q}{A_o A_m} \text{ is SPR and } \phi \text{ signals are bounded}$$

• اطلاع از b_0 برای محاسبه ε لازم است.

• اگر b_0 نامعلوم باشد می توان آن را تخمیین زد:

$$e_f = \frac{b_0 Q}{A_o A_m}\left(\frac{1}{P_1}u + \phi^T \theta^0\right)$$

$$= b_0\left(\frac{Q}{A_o A_m}\frac{1}{P_1}u + \frac{Q}{A_o A_m}\phi^T \theta^0\right)$$

تعریف کنید:

$$e_f = b_0\left(u_f + \phi_f^{\,T} \theta^0\right)$$

تخمیین زننده های ساده گرادیانی

$$\frac{d\theta}{dt} = \gamma' \hat{b}_0 \phi_f \varepsilon_P = \gamma \phi_f \varepsilon_P$$

خطای پیش بینی

$$\frac{d\hat{b}_0}{dt} = \gamma(\phi^T{}_f \theta + u_f)\varepsilon_P$$

که در آن

$$\varepsilon_f = e_f - \hat{e}_f = e_f - \hat{b}_0\left(u_f + \phi_f^{\,T} \theta\right)$$

• تحقق MRAS

The equations needed to implement the general MRAS can now be summarized:

$$y_m = \frac{B_m}{A_m} u_c$$

$$e_f = \frac{Q}{P} e = \frac{Q}{P} (y - y_m)$$

$$\eta = -\left(\frac{1}{P_1} u + \varphi^T \theta\right)$$

$$\varepsilon = e_f + \frac{b_0 Q}{A_o A_m} \eta$$

$$\frac{d\theta}{dt} = \gamma \varphi \varepsilon$$

$$u = -\theta^T (P_1 \varphi)$$

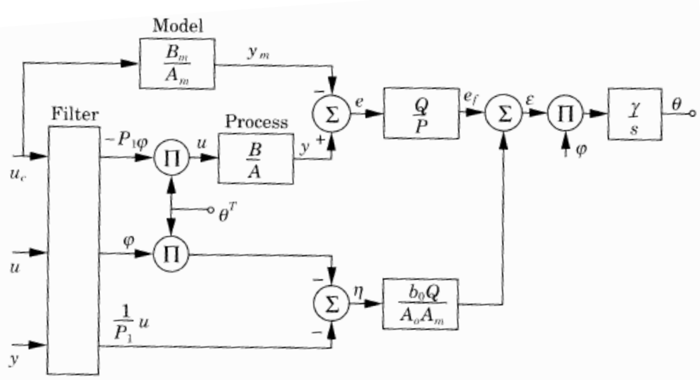


Figure 5.21 Block diagram of a model-reference adaptive system for a SISO system.

having the same structure. It therefore suffices to discuss one part. Consider, for example, how to generate φ_u and $P_1\varphi_u$ where

$$P_1\varphi_u = \left(\frac{p^{k-1}}{P_2}u \dots \frac{1}{P_2}u\right)^T = (x_1 \dots x_k)^T = x^T$$

and

$$\varphi_u = \left(\frac{p^{k-1}}{P}u \dots \frac{1}{P}u\right)^T$$

where $P = P_1P_2$ and $k = \deg R = \deg P_2$.

Let the polynomials P_1 and P_2 be

$$P_1 = p^n + \alpha_1 p^{n-1} + \dots + \alpha_n$$
$$P_2 = p^k + \beta_1 p^{k-1} + \dots + \beta_k$$

We also assume that $\deg P_1 > \deg P_2$. The vectors x and φ_u can then be realized as follows:

$$\frac{dx}{dt} = \begin{pmatrix} -\beta_1 & -\beta_2 & \dots & -\beta_{k-1} & -\beta_k \\ 1 & 0 & & 0 & 0 \\ \vdots & \ddots & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$\frac{dz}{dt} = \begin{pmatrix} -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & & 0 & 0 \\ \vdots & \ddots & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} x_k$$

where $x_k = 1/P_2 \cdot u$ is the last element of the x vector. The elements of φ_u are the k last elements of the state vector z. Furthermore, $1/P_1 \cdot u$ can also be obtained from the generation of φ_u and $P_1\varphi_u$. To generate the full vectors φ and $P_1\varphi$, we thus need three realizations of the transfer functions P_1 and P_2. The block labeled "Filter" in Fig. 5.21 represents these systems.

- پارامترهای طراحی
  - ✓ مدل مرجع
  - ✓ چندجمله ای رویتگر
  - ✓ درجه چند جمله ای های رگلاتور
  - ✓ چند جمله ای های Q, P_1, P_2
- اطلاعات قبلی لازم
  - ✓ علامت b_0
  - ✓ می نیمم فاز بودن سیستم
  - ✓ درجه نسبی سیستم
  - ✓ مرتبه سیستم

Compare STR and MRAS

MRAS

$$\frac{d\theta}{dt} = \gamma \varphi_f \varepsilon$$

$$\varphi_f^T = -G_f(p)\mathrm{grad}_\theta \varepsilon(t)$$

$$\varepsilon = G_{SPR}(y - y_m) + \eta = G_{SPR}e + \eta$$

Direct STR

$$y(t) = \varphi_f^T(t - d_0)\,\theta$$

$$\varepsilon(t) = y(t) - \hat{=} y(t) - \varphi_f^T(t - d_0)\,\hat\theta(t-1)$$

$$\hat\theta(t) = \hat\theta(t-1) + P(t)\varphi_f^T(t - d_0)\,\varepsilon(t)$$

Residual

$$\varepsilon(t) = y(t) - \hat{y}(t) = y(t) - y_m(t) + y_m(t) - \hat{y}(t)$$
$$= e(t) + \eta(t)$$

❖ نتیجه گیری

- ✓ آشنایی با ایده های MRAS
- ✓ روش گرادیان در طراحی MRAS
- ✓ روش لیاپانوف در طراحی MRAS پایدار
- ✓ ایده پسیویتی در طراحی MRASپایدار
- ✓ خطای افزوده شده
- ✓ مقایسه MRAS و STR