



Analytical approach to tuning of model predictive control for first-order plus dead time models

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Abstract: Model predictive control (MPC) is an effective control strategy in the presence of system constraints. The successful implementation of MPC in practical applications requires appropriate tuning of the controller parameters. An analytical tuning strategy for MPC of first-order plus dead time (FOPDT) systems is presented when the constraints are inactive. The available tuning methods are generally based on the user's experience and experimental results. Some tuning methods lead to a complex optimisation problem that provides numerical results for the controller parameters. On the other hand, many industrial plants can be effectively described by FOPDT models, and this model is therefore used to derive analytical results for the MPC tuning in a pole placement framework. Then, the issues of closed-loop stability and possible achievable performance are addressed. In the case of no active constraints, it is shown that for the FOPDT models, control horizons subsequent to two do not improve the achievable performance and control horizon of two provides the maximum achievable performance. Then, MPC tuning for higher order plants approximated by FOPDT models is considered. Finally, simulation results are employed to show the effectiveness of the proposed tuning formulas.

1 Introduction

Model predictive control (MPC) is widely used in many advanced process control systems [1–3]. The ability of MPC in constraint handling is well known, and this capability makes it more effective in real applications [4].

MPC is a model-based control strategy and model complexity substantially increases its computational burden, which is a key deterrent factor in many practical applications. However, many industrial processes can be sufficiently described by first-order plus dead time (FOPDT) models [5]. Hence, this model is used to analyse the MPC closed-loop behaviour and leads to effective tuning formulas.

MPC has several tuning parameters that must be appropriately tuned for a competent closed-loop control in practical applications. For a typical MPC, conventional tuning parameters include the prediction and control horizons and the weighted matrices used in the cost function. These parameters can significantly influence the closed-loop performance, stability and robustness characteristics and are therefore extensively studied in many research papers [6, 7]. However, owing to the complex inter-relations between the MPC and the process parameters and a desired closed-loop stability and performance characteristics, the tuning procedure is an intricate problem and active constraints substantially complicates this problem to the point that only trial and error methods are available for MPC tuning. In a general classification for no active constraints, different tuning approaches can be grouped as the analytical-based

and the numerical-based approaches. Owing to the problem complexity, there are limited analytical results in this area. Analytical equations for tuning parameters of dynamic matrix control (DMC) based on the FOPDT model approximation can be found in [8]. In [8], a weighting factor is tuned to avoid singularity in the control signal calculation, but closed-loop performance is not considered in this formulation. In [9], by using robust performance number, a tuning procedure for MPC is developed, which is applicable to multivariable non-minimum phase plants. An analytical formulation for DMC tuning using some practical approaches is presented in [10]. However, based on the results in [8], the formula can lead to erroneous results for different open-loop dc gains. An extension of the modified generalised predictive control algorithm and a tuning strategy for the plants described by the second-order plus dead time (SOPDT) models are developed in [11]. Tuning equations for DMC parameters are developed in [12] based on the application of analysis of variance and non-linear regression analysis for FOPDT process models. Although these results provide closed-form solutions for the tuning problem, they are not based on rigorous mathematical analysis. In [13], a MPC tuning method is proposed to achieve closed-loop robust performance based on state estimation and sensitivity functions, where choosing a large enough control horizon is suggested. There are several results available on MPC tuning based on numerical optimisation techniques. An online DMC tuning methodology is presented in [14] based on a constrained least square optimisation that tunes parameters to satisfy

a predefined closed-loop time domain performance. A tuning procedure to achieve desired closed-loop performance is proposed in [15], where the weight matrices and control horizon are tuned using a convex optimisation approach. It is assumed that the control horizon is fixed to one, as a constraint to solve the optimisation problem. In [16], an inverse-engineering-based methodology is used to find the cost function and state estimator used in MPC. This is subsequently extended in [17] to give a new tuning method for MPC. In [18], two methods for selecting the MPC weight matrices that result in linear state feedback controllers are derived, where multivariable controllers are considered and linear matrix inequality (LMI) methods solve the inverse problem of the controller matching which numerically tunes the weight matrices. Finally, in a recent research [19], a tuning strategy for multi-parametric predictive controller is developed. The controller tuning is based on local linear analysis of the closed-loop system.

In this paper, a new analytical method for MPC tuning is proposed when the constraints are inactive. Stable FOPDT models are considered and closed-loop transfer functions are obtained. Then, the tuning problem is restated as a pole placement problem. To derive exact tuning formulas with guaranteed closed-loop stability and desired performance specifications, control horizons of one and two are considered, respectively. In addition, simulation studies are performed to further study the closed-loop properties with control horizons greater than two. In the case of no active constraints, analytical tuning equations are obtained and achievable performance for the closed-loop system is addressed. It is shown that the closed-loop plant with control horizon of two can achieve the maximum achievable desired performance. That is, the MPC performance for FOPDT plants does not improve by increasing the control horizon. This performance is defined as the feasible performance and helps to maintain the least computational cost by avoiding higher control horizons. Higher order plants approximated by FOPDT models are also considered. Finally, simulation results are used to show the effectiveness of the proposed tuning methodology for the FOPDT and higher order plants. The paper is organised as follows: In Section 2, the state-space MPC formulation for FOPDT plants is given and closed-loop transfer functions are obtained. Higher order plants described by FOPDT models and DMC for FOPDT models are also considered. Section 3 provides the analytical MPC tuning equations. The efficiency of the proposed tuning algorithms is analysed through examples in Section 4. Finally, Section 5 ends the paper with concluding results.

2 MPC formulation for FOPDT models

In this section, a standard structure for constrained MPC of FOPDT models is given. The state-space model is considered and in the case of no active constraints, the closed-loop transfer function is obtained. Finally, the closed-loop transfer function in the case of higher order plants described by FOPDT models is considered.

2.1 State-space MPC formulation for FOPDT models

Consider the following stable FOPDT model

$$G_m(s) = \frac{k_p e^{-\theta s}}{\tau s + 1} \quad (1)$$

and the corresponding discrete time model with a sampling time T_s is

$$G_m(z^{-1}) = \frac{k_p(1-a)z^{-k-1}}{1-az^{-1}} \quad (2)$$

where $a = e^{-T_s/\tau}$; the dead time is considered to be an integer multiple of the sampling time, that is, $k = \theta/T_s$. The augmented state-space model with integrator is as follows

$$\begin{aligned} \mathbf{x}(n+1) &= \mathbf{A}\mathbf{x}(n) + \mathbf{B}\Delta u(n) \\ y_m(n) &= \mathbf{C}\mathbf{x}(n) \end{aligned} \quad (3)$$

where $\Delta = 1 - z^{-1}$ and

$$\begin{aligned} \mathbf{x}(n) &= [\Delta y_m(n) \quad \Delta y_m(n+1) \quad \cdots \quad \Delta y_m(n+k) \quad y_m(n+k)]^T \\ \mathbf{A} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \hline 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & a & 1 \end{bmatrix}, \quad \mathbf{B} = k_p(1-a) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ \mathbf{C} &= [0 \quad -1 \quad -1 \quad \cdots \quad -1 \quad 1] \end{aligned} \quad (4)$$

The above realisation is both controllable and observable. The finite optimal control problem is as follows

$$\begin{aligned} \min_{u(n)} & (\mathbf{w}(n) - \mathbf{y}(n))^T \mathbf{Q}(\mathbf{w}(n) - \mathbf{y}(n)) + (\Delta \mathbf{u}(n))^T \mathbf{R}(\Delta \mathbf{u}(n)) \\ \text{s.t. } & u_{\min} \leq u(n+i|n) \leq u_{\max}, \quad i = 0, 1, \dots, M-1 \\ & y_{\min} \leq \hat{y}(n+j|n) \leq y_{\max}, \quad j = N_1, N_1+1, \dots, N_2 \end{aligned} \quad (5)$$

where

$$\begin{aligned} \mathbf{w}(n) &= \begin{bmatrix} w(n) \\ w(n) \\ \vdots \\ w(n) \end{bmatrix}_{P \times 1}, \quad \mathbf{y}(n) = \begin{bmatrix} \hat{y}(n+N_1|n) \\ \hat{y}(n+N_1+1|n) \\ \vdots \\ \hat{y}(n+N_2|n) \end{bmatrix}_{P \times 1}, \\ \Delta \mathbf{u}(n) &= \begin{bmatrix} \Delta u(n) \\ \Delta u(n+1) \\ \vdots \\ \Delta u(n+M-1) \end{bmatrix}_{M \times 1} \\ \mathbf{Q} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & q_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q_P \end{bmatrix}_{P \times P}, \\ \mathbf{R} &= k_p^2(1-a)^2 \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_M \end{bmatrix}_{M \times M} \end{aligned} \quad (6)$$

Note that $N_1 = k+1$, $N_2 = k+P$, where P is the prediction horizon, M is the control horizon and $\hat{y}(\cdot|n)$ is the predicted value of plant output at instance n and the weight on control

effort is normalised. Denote the future output values as

$$y(n) = Fx(n) + S\Delta u(n) \quad (7)$$

where

$$F = \begin{bmatrix} CA^{k+1} \\ CA^{k+2} \\ \vdots \\ CA^{k+P} \end{bmatrix}_{P \times (k+2)}, \quad S = \begin{bmatrix} CA^k B & 0 & 0 & 0 \\ CA^{k+1} B & CA^k B & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{k+P-1} B & CA^{k+P-2} B & \dots & CA^{k+P-M} B \end{bmatrix}_{P \times M} \quad (8)$$

According to (4) and (8), we have

$$F = [F_1 \ F_2 \ \dots \ F_k \ F_{k+1} \ F_{k+2}]$$

$$F_1 = F_2 = \dots = F_k = \mathbf{0}_{P \times 1},$$

$$F_{k+1} = a \begin{bmatrix} 1 \\ 1+a \\ \vdots \\ 1+a+\dots+a^{P-2}+a^{P-1} \end{bmatrix}_{P \times 1}, \quad F_{k+2} = \mathbf{I}_{P \times 1}$$

(see (9))

where $\mathbf{I}_{P \times 1} = [1 \ 1 \ \dots \ 1]^T$. In the case of no active constraints, the optimal control effort solution is

$$\Delta u(n) = (R + S^T Q S)^{-1} S^T Q (I_{P \times 1} w(n) - F_{k+1} \Delta \hat{y}(n+k) - I_{P \times 1} \hat{y}(n+k)) \quad (10)$$

where $\hat{y}(n+k)$ is the real output prediction.

$$\hat{y}(n+k) = a^k y_m(n) + k_p(1-a)(a^{k-1}u(n-k) + \dots + au(n-2) + u(n-1)) + d(n) \quad (11)$$

where

$$d(n) = y_p(n) - y_m(n) \quad (12)$$

where $y_p(\cdot)$ is the plant output. Mathematical operations on (10) lead to

$$\Delta u(n) = K_y (w(n) - \hat{y}(n+k)) - K_x \Delta \hat{y}(n+k) \quad (13)$$

where

$$K_x = \begin{bmatrix} K_{x1} \\ K_{x2} \\ \vdots \\ K_{xM} \end{bmatrix} = (R + S^T Q S)^{-1} S^T Q F_{k+1},$$

$$K_y = \begin{bmatrix} K_{y1} \\ K_{y2} \\ \vdots \\ K_{yM} \end{bmatrix} = (R + S^T Q S)^{-1} S^T Q I_{P \times 1} \quad (14)$$

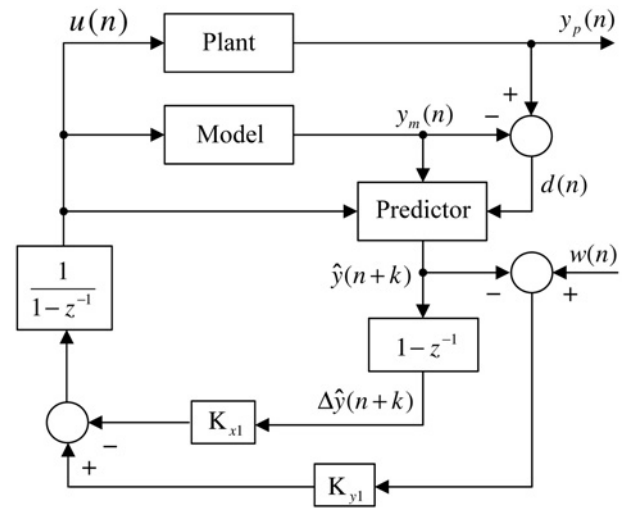


Fig. 1 Block diagram of the closed-loop plant

Now, the current control effort is obtained as follows

$$\Delta u(n) = K_{y1}(w(n) - \hat{y}(n+k)) - K_{x1}\Delta \hat{y}(n+k) \quad (15)$$

The block diagram of the proposed control configuration is shown in Fig. 1.

2.1.1 Closed-loop plant analysis: Let $d(n) = 0$, that is the plant and model outputs are the same. Using (2), (11), (12) and (15), it can be shown that the closed-loop transfer function is

$$G_{cl}(z) = \frac{K'_{y1}}{z^{k-1}[z^2 + z(-1-a+K'_{x1}+K'_{y1}) + (a-K'_{x1})]} \quad (16)$$

where

$$K'_{x1} = k_p(1-a)K_{x1}, \quad K'_{y1} = k_p(1-a)K_{y1} \quad (17)$$

Note that in the case of no model mismatch and no active constraints, MPC of the FOPDT model (2) is a SOPDT transfer function.

Lemma 1: The closed-loop plant (16) is stable if

$$K'_{y1} > 0, \quad 2(1+a-K'_{x1}) > K'_{y1}, \quad a-1 < K'_{x1} < a+1 \quad (18)$$

Proof: Direct application of the Jury's test [20] proves the result. The stability plot of K'_{y1} against K'_{x1} is illustrated in Fig. 2.

Note that the desired closed-loop performance is achievable by properly selecting the gains K'_{x1} and K'_{y1} . \square

$$S = k_p(1-a) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1+a & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1+a+\dots+a^{P-1} & 1+a+\dots+a^{P-2} & \dots & 1+a+\dots+a^{P-M} \end{bmatrix}_{P \times M} \quad (9)$$

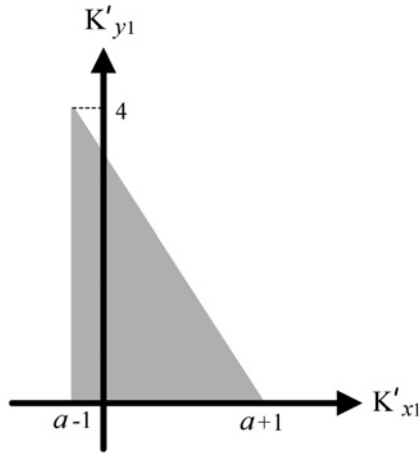


Fig. 2 Stability region for K'_{x1} and K'_{y1}

Note 1: In the case of higher order plants, the real plant can be described as follows

$$G_p(z^{-1}) = \frac{y_p(n)}{u(n)} = \frac{N(z^{-1})}{D(z^{-1})} \quad (19)$$

and the model employed for the predictive control design is

$$G_m(z^{-1}) = \frac{y_m(n)}{u(n)} = \frac{k_p(1-a)z^{-k-1}}{1-az^{-1}} \quad (20)$$

Equation (11) gives

$$\hat{y}(n+k) = a^k y_m(n) + k_p(1-a)z^{-1} \left(\frac{1-a^k z^{-k}}{1-az^{-1}} \right) u(n) + d(n) \quad (21)$$

we have

$$d(n) = y_p(n) - y_m(n) = \left(\frac{N(z^{-1})}{D(z^{-1})} - \frac{k_p(1-a)z^{-k-1}}{1-az^{-1}} \right) u(n) \quad (22)$$

which gives

$$\hat{y}(n+k) = y_p(n) \left(1 + k_p(1-a) \frac{z^{-1} - z^{-k-1}}{1-az^{-1}} \frac{D(z^{-1})}{N(z^{-1})} \right) \quad (23)$$

Now (15) and (23) leads to

$$\begin{aligned} \Delta u(n) &= K_{y1} w(n) \\ &+ y_p(n) \left(1 + k_p(1-a) \frac{z^{-1} - z^{-k-1}}{1-az^{-1}} \frac{D(z^{-1})}{N(z^{-1})} \right) \\ &\times (K_{y1} + (1-z^{-1})K_{x1}) \end{aligned} \quad (24)$$

Finally, the closed-loop transfer function is derived as follows

$$G_{cl}(z) = \frac{K'_{y1}(1-az^{-1})N(z^{-1})}{\Delta_{cl}(z^{-1})} \quad (25)$$

where

$$\begin{aligned} \Delta_{cl}(z^{-1}) &= k_p(1-a)(1-z^{-1})(1-az^{-1})D(z^{-1}) \\ &+ \{(1-az^{-1})N(z^{-1}) \\ &+ k_p(1-a)D(z^{-1})(z^{-1} - z^{-k-1})\} \\ &\times (K'_{y1} + (1-z^{-1})K'_{x1}) \end{aligned} \quad (26)$$

The closed-loop transfer functions (16) and (25) developed in the above sections are used to derive exact tuning formulas for MPC of FOPDT models in the next section.

Note 2: In the case of DMC, we have

$$\Delta u(n) = \sum_{i=k+1}^{k+P} c_i (w(n+i) - f(n+i)) \quad (27)$$

where c_i denotes the i th first row elements of the matrix $(G^T G + \lambda I)^{-1} G^T$, G is the dynamic matrix and λ is the weighting on the control effort. Also, f is the free response part of the output prediction. We have

$$f(n+j) = y_p(n) + \sum_{j=1}^{N-1} c_i (g_{i+j} - g_j) \Delta u(n-j) \quad (28)$$

where N is the model horizon and g_i is the i th unit step response coefficient of the model

$$\begin{aligned} \Delta u(n) &= \sum_{i=k+1}^{k+P} c_i (w(n) - y_p(n)) \\ &- \sum_{i=k+1}^{k+P} c_i \sum_{j=1}^{N-1} (g_{i+j} - g_j) \Delta u(n-j) \end{aligned} \quad (29)$$

which gives

$$\Delta u(n) \left(1 + \sum_{i=k+1}^{k+P} c_i \sum_{j=1}^{N-1} (g_{i+j} - g_j) z^{-j} \right) = \sum_{i=k+1}^{k+P} c_i e(n) \quad (30)$$

where $e(n) = w(n) - y_p(n)$. The DMC transfer function is

$$\frac{u(n)}{e(n)} = \frac{\sum_{i=k+1}^{k+P} c_i}{(1-z^{-1})(1 + \sum_{i=k+1}^{k+P} c_i \sum_{j=1}^{N-1} (g_{i+j} - g_j) z^{-j})} \quad (31)$$

which yields the following closed-loop transfer function

$$G_{cl}(z) = \frac{k_p(1-a)z^{-k-1} \sum_{i=k+1}^{k+P} c_i}{\Delta_{cl}(z^{-1})} \quad (32)$$

where

$$\begin{aligned} \Delta_{cl}(z^{-1}) &= (1-az^{-1})(1-z^{-1}) \left(1 + \sum_{i=k+1}^{k+P} c_i \sum_{j=1}^{N-1} (g_{i+j} - g_j) z^{-j} \right) \\ &+ k_p(1-a)z^{-k-1} \sum_{i=k+1}^{k+P} c_i \end{aligned} \quad (33)$$

For the unit step response coefficients of the FOPDT model (2), we have

$$g_j = \begin{cases} k_p(1-a^{j-k}), & j > k \\ 0, & j \leq k \end{cases} \quad (34)$$

Several mathematical manipulations yield (see Appendix 1)

$$\begin{aligned} \Delta_{cl}(z^{-1}) &= 1 + z^{-1} \left(-1 - a + k_p \sum_{i=k+1}^{k+P} c_i - k_p a^{1-k} \sum_{i=k+1}^{k+P} c_i a^i \right) \\ &\quad + z^{-2} \left(a - k_p a \sum_{i=k+1}^{k+P} c_i + k_p a^{1-k} \sum_{i=k+1}^{k+P} c_i a^i \right) \end{aligned} \quad (35)$$

Comparing (32), (35), (16) and (17) leads to

$$\begin{aligned} K'_{y1} &= k_p(1-a) \sum_{i=k+1}^{k+P} c_i \\ K'_{x1} &= k_p a \sum_{i=k+1}^{k+P} c_i - k_p a^{1-k} \sum_{i=k+1}^{k+P} c_i a^i \end{aligned} \quad (36)$$

So

$$\begin{aligned} \sum_{i=k+1}^{k+P} c_i &= \text{First element of } \{(\mathbf{R} + \mathbf{S}^T \mathbf{Q} \mathbf{S})^{-1} \mathbf{S}^T \mathbf{Q} \mathbf{I}_{P \times 1}\} \\ \frac{a \sum_{i=k+1}^{k+P} c_i (1 - a^{i-k})}{1 - a} &= \text{First element of } \{(\mathbf{R} + \mathbf{S}^T \mathbf{Q} \mathbf{S})^{-1} \mathbf{S}^T \mathbf{Q} \mathbf{F}_{k+1}\} \end{aligned} \quad (37)$$

Note 3: The structure of a predictive proportional-integral (PI) controller, where the Smith predictor is combined with the PI controller, which is given in [21] and is similar to the proposed MPC formulation for the FOPDT models. The results proved in Section 2.1 are the backbones for derivation of MPC analytical tuning formulations for FOPDT models. However, it can be shown that based on the results of [21] the closed-loop transfer function for FOPDT model is

$$G_{cl}(z) = \frac{z(K'_{x1} + K'_{y1}) - K'_{x1}}{z^k [z^2 + z(-1 - a + K'_{x1} + K'_{y1}) + (a - K'_{x1})]}$$

which is similar to (16).

3 Tuning formulas for the MPC

In this section, when there are no active constraints the tuning formulas for the MPC are derived and the resulted control structure is analysed. The tuning formulas are derived for the two separate cases of control horizons of $M = 1$ and $M = 2$.

Using (16), it is possible to place the closed-loop poles in desired positions by appropriately choosing the desired gains K'_{xd1} and K'_{yd1} . This leads to analytical equations for tuning parameters and determines the possible attainable performance for the MPC of FOPDT plants. It is obvious from the closed-loop transfer function (16) that the desired gains K'_{xd1} and K'_{yd1} can lead to the desired performance. However, it is important to note that not any desired performance is feasible. In what follows, the key feasibility concept is considered and relevant theorems are introduced.

The tuning parameters are \mathbf{Q} , \mathbf{R} , P and M , where \mathbf{Q} is a diagonal positive semi-definite matrix, \mathbf{R} is a diagonal positive-definite matrix, P and M are natural numbers, and the sampling time is assumed fixed and appropriately chosen.

Definition 1 (Feasible gains): The desired gains K'_{xd1} and K'_{yd1} that satisfy (14) and (17) are called the feasible gains.

Note that the gains are selected prior to (14) and (17). If these gains are feasible, then there exists \mathbf{Q} , \mathbf{R} , P and M that satisfy (14) and subsequently (17). However, there may not exist such parameters, in which case the gains would be infeasible.

3.1 Control horizon of one

Consider the control horizon of one in (14). It is easily shown that

$$\begin{aligned} K_{x1} &= (\mathbf{R} + \mathbf{S}^T \mathbf{Q} \mathbf{S})^{-1} \mathbf{S}^T \mathbf{Q} \mathbf{F}_{k+1} = \frac{1}{k_p(1-a)} \frac{aX}{X+r} \\ K_{y1} &= (\mathbf{R} + \mathbf{S}^T \mathbf{Q} \mathbf{S})^{-1} \mathbf{S}^T \mathbf{Q} \mathbf{I}_{P \times 1} = \frac{1}{k_p(1-a)} \frac{Y}{X+r} \end{aligned} \quad (38)$$

where

$$\begin{aligned} X &= 1 + q_2(1+a)^2 + q_3(1+a+a^2)^2 + \dots \\ &\quad + q_P(1+a+\dots+a^{P-1})^2 \\ Y &= 1 + q_2(1+a) + q_3(1+a+a^2) + \dots \\ &\quad + q_P(1+a+\dots+a^{P-1}) \end{aligned} \quad (39)$$

Let K'_{xd1} and K'_{yd1} defined in (17) be the desired gains. Note that these gains are attainable by tuning only two parameters. It is known that in MPC tuning problem, the weight matrices are more dominant on system performance than other tuning parameters [8]. So, the weights on the cost function (5), that is, q_2, q_3, \dots, q_P and r are chosen as the tuning parameters.

Theorem 1: Let

$$\mathbf{Q} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & q_P \end{bmatrix}, \quad \mathbf{R} = k_p^2(1-a)^2 r \quad (40)$$

where \mathbf{I} and $\mathbf{0}$ have proper dimensions. In this case, the desired feasible gains K'_{xd1} and K'_{yd1} for control horizon of one satisfy the following inequalities

$$0 < K'_{yd1}, \quad 0 < K'_{xd1} < a, \quad \frac{1}{a} \frac{1}{X_1} < \frac{K'_{yd1}}{K'_{xd1}} < \frac{1}{a} \frac{Y_2}{X_2} \quad (41)$$

By selecting q_P and r as the tuning parameters, the tuning equations for achieving these gains are

$$\begin{aligned} q_P &= \frac{Y_2 K'_{xd1} - a X_2 K'_{yd1}}{X_1 (a X_1 K'_{yd1} - K'_{xd1})} \\ r &= \frac{a - K'_{xd1}}{K'_{xd1}} (X_2 + q_P X_1^2) \end{aligned} \quad (42)$$

where

$$\begin{aligned} X_1 &= 1 + a + \dots + a^{P-1} \\ X_2 &= 1 + (1+a)^2 + (1+a+a^2)^2 + \dots \\ &\quad + (1+a+\dots+a^{P-2})^2 \\ Y_2 &= 1 + (1+a) + (1+a+a^2) + \dots \\ &\quad + (1+a+\dots+a^{P-2}) \end{aligned} \quad (43)$$

Proof: Equations (17), (38) and (39) lead to

$$r = \frac{a - K'_{x1}}{K'_{x1}}(X_2 + q_P X_1^2)$$

$$a(X_2 + q_P X_1^2)K'_{y1} = (Y_2 + q_P X_1)K'_{x1}$$

$$\rightarrow q_P = \frac{Y_2 K'_{x1} - a X_2 K'_{y1}}{X_1(a X_1 K'_{y1} - K'_{x1})}$$

Note that r should be positive, so

$$\frac{a - K'_{x1}}{K'_{x1}} > 0 \xrightarrow{0 < a < 1} 0 < K'_{x1} < a$$

and q_P should be positive, therefore

$$\frac{Y_2 K'_{x1} - a X_2 K'_{y1}}{X_1(a X_1 K'_{y1} - K'_{x1})} > 0 \rightarrow Y_2 - \frac{K'_{y1}}{K'_{x1}} a X_2 > 0,$$

$$\frac{K'_{y1}}{K'_{x1}} a X_1 - 1 > 0 \rightarrow \frac{1}{a} \frac{1}{X_1} < \frac{K'_{y1}}{K'_{x1}} < \frac{1}{a} \frac{Y_2}{X_2}$$

□

Remark 1: The lower and upper limits of (K'_{y1}/K'_{x1}) determine the possible tuning region. Increasing the prediction horizon, this region gets smaller.

Theorem 2: Consider the MPC problem with two coincidence points $N_1 = k + 1$ and $N_2 = k + P$, that gives

$$\mathbf{Q} = \begin{bmatrix} 1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \\ & & & & q_P \end{bmatrix} \quad (44)$$

In this case, the desired feasible gains K'_{xd1} and K'_{yd1} satisfy the following inequalities

$$0 < K'_{yd1}, \quad 0 < K'_{xd1} < a, \quad \frac{1}{a} \frac{1}{X_1} < \frac{K'_{yd1}}{K'_{xd1}} < \frac{1}{a} \quad (45)$$

and the tuning equations for achieving these gains are

$$q_P = \frac{K'_{xd1} - a K'_{yd1}}{X_1(K'_{yd1} a X_1 - K'_{xd1})}$$

$$r = \frac{a - K'_{xd1}}{K'_{xd1}}(1 + q_P X_1^2) \quad (46)$$

where X_1 is as defined in (43).

Proof: Equations (38), (39) and (44) give

$$r = \frac{a - K'_{x1}}{K'_{x1}}(1 + q_P X_1^2)$$

$$a(X_2 + q_P X_1^2)K'_{y1} = (1 + q_P X_1)K'_{x1}$$

$$\rightarrow q_P = \frac{K'_{x1} - a K'_{y1}}{X_1(a X_1 K'_{y1} - K'_{x1})}$$

The positivity of r and q_P easily show the results. □

Corollary 1: In the case of infinite prediction horizon in Theorem 2, the maximum area of feasible gains is given by the following inequalities

$$0 < K'_{yd1}, \quad 0 < K'_{xd1} < a, \quad \frac{1 - a}{a} < \frac{K'_{yd1}}{K'_{xd1}} < \frac{1}{a} \quad (47)$$

Proof: Equations (43) and (45) and the assumption of $P \rightarrow \infty$ completes the proof. □

Remark 2: In Theorem 2, any other choice for coincidence points leads to smaller feasibility areas.

3.2 Control horizon of two

In this section, the control horizon of two in (10) is considered, where the weights on control efforts are the selected tuning parameters. We have

$$\mathbf{K}_x = \begin{bmatrix} K_{x1} \\ K_{x2} \end{bmatrix} = (\mathbf{R} + \mathbf{S}^T \mathbf{Q} \mathbf{S})^{-1} \mathbf{S}^T \mathbf{Q} \mathbf{F}_{k+1},$$

$$\mathbf{K}_y = \begin{bmatrix} K_{y1} \\ K_{y2} \end{bmatrix} = (\mathbf{R} + \mathbf{S}^T \mathbf{Q} \mathbf{S})^{-1} \mathbf{S}^T \mathbf{Q} \mathbf{I}_{p \times 1} \quad (48)$$

Using (17) and (48) yields

$$\begin{bmatrix} X_{11} + r_1 & X_{12} \\ X_{12} & X_{22} + r_2 \end{bmatrix} \begin{bmatrix} K'_{x1} & K'_{y1} \\ K'_{x2} & K'_{y2} \end{bmatrix} = \begin{bmatrix} a X_{11} & Y_{11} \\ a X_{12} & Y_{22} \end{bmatrix} \quad (49)$$

where

$$X_{11} = 1 + q_2(1 + a)^2 + q_3(1 + a + a^2)^2 + \dots$$

$$+ q_P(1 + a + \dots + a^{P-1})^2$$

$$X_{12} = q_2(1 + a) + q_3(1 + a)(1 + a + a^2) + \dots$$

$$+ q_P(1 + \dots + a^{P-2})(1 + \dots + a^{P-1})$$

$$X_{22} = q_2 + q_3(1 + a)^2 + q_4(1 + a + a^2)^2 + \dots$$

$$+ q_P(1 + a + \dots + a^{P-2})^2$$

$$Y_{11} = 1 + q_2(1 + a) + q_3(1 + a + a^2) + \dots$$

$$+ q_P(1 + a + \dots + a^{P-1})$$

$$Y_{22} = q_2 + q_3(1 + a) + q_4(1 + a + a^2) + \dots$$

$$+ q_P(1 + \dots + a^{P-2}) \quad (50)$$

Theorem 3: In the case of control horizon two, and considering the prediction error weights as arbitrary and the weights on control efforts as the tuning parameters, the desired feasible gains K'_{xd1} and K'_{yd1} satisfy the following inequalities

$$0 < K'_{yd1}, \quad 0 < K'_{xd1} < a, \quad \frac{1}{a} \frac{Y_{11}}{X_{11}} < \frac{K'_{yd1}}{K'_{xd1}} < \frac{1}{a} \quad (51)$$

and the tuning equations for achieving these gains are

$$r_1 = \frac{(Y_{11} X_{12} - X_{11} Y_{22})(a - K'_{xd1})}{a X_{12} K'_{yd1} - Y_{22} K'_{xd1}},$$

$$r_2 = \frac{(X_{22} Y_{11} - X_{12} Y_{22})(K'_{xd1} - a K'_{yd1})}{a X_{11} K'_{yd1} - Y_{11} K'_{xd1}} \quad (52)$$

Proof: Equation (49) gives

$$\begin{aligned} r_1 &= \frac{1}{K'_{x1}} (aX_{11} - X_{11}K'_{x1} - X_{12}K'_{x2}) \\ &= \frac{1}{K'_{y1}} (Y_{11} - X_{11}K'_{y1} - X_{12}K'_{y2}) \\ r_2 &= \frac{1}{K'_{x2}} (aX_{12} - X_{11}K'_{x1} - X_{22}K'_{x2}) \\ &= \frac{1}{K'_{y2}} (Y_{22} - X_{12}K'_{y1} - X_{22}K'_{y2}) \end{aligned} \quad (53)$$

so

$$\begin{aligned} aX_{11}K'_{y1} - Y_{11}K'_{x1} &= X_{12}(K'_{x2}K'_{y1} - K'_{y2}K'_{x1}) \\ Y_{22}K'_{x2} - aX_{12}K'_{y2} &= X_{12}(K'_{x2}K'_{y1} - K'_{y2}K'_{x1}) \end{aligned}$$

Now K'_{y2} is calculated using the above equations as

$$K'_{y2} = \frac{aX_{11}Y_{22}K'_{y1} - Y_{11}Y_{22}K'_{x1} + Y_{11}X_{12}K'_{x1}K'_{y1} - aX_{11}X_{12}K'_{y1}^2}{X_{12}(aX_{12}K'_{y1} - Y_{22}K'_{x1})}$$

Then, the tuning equations for r_1 and r_2 are obtained by manipulating (53) as

$$\begin{aligned} r_1 &= \frac{(X_{12}Y_{11} - X_{11}Y_{22})(a - K'_{x1})}{aX_{12}K'_{y1} - Y_{22}K'_{x1}} \\ r_2 &= \frac{-K'_{x1}(X_{12}Y_{22} - X_{22}Y_{11}) + aK'_{y1}(X_{11}X_{22} - X_{12}^2)}{Y_{11}K'_{x1} - aX_{11}K'_{y1}} \end{aligned}$$

It can be shown that $X_{22}Y_{11} - X_{12}Y_{22} = X_{11}X_{22} - X_{12}^2$ (see Appendix 1), so the tuning equation for r_2 can be rewritten as

$$r_2 = \frac{(X_{22}Y_{11} - X_{12}Y_{22})(K'_{x1} - aK'_{y1})}{aX_{11}K'_{y1} - Y_{11}K'_{x1}}$$

As \mathbf{R} is a positive-definite matrix, we have from (52)

$$r_1 = \frac{(Y_{11}X_{12} - X_{11}Y_{22})(a - K'_{x1})}{aX_{12}K'_{y1} - Y_{22}K'_{x1}} > 0 \quad (54)$$

Using mathematical induction, we can show that $X_{12}Y_{11} - X_{11}Y_{22} > 0$ (for more details see Appendix 1). Thus, the following sets of inequalities satisfy (54)

$$\begin{aligned} 0 < K'_{x1} < a, \quad \frac{Y_{22}}{aX_{12}} < \frac{K'_{y1}}{K'_{x1}} \text{ or } K'_{x1} < 0, \\ \frac{K'_{y1}}{K'_{x1}} < -\frac{Y_{22}}{aX_{12}} \text{ or } a < K'_{x1}, \quad \frac{K'_{y1}}{K'_{x1}} < \frac{Y_{22}}{aX_{12}} \end{aligned} \quad (55)$$

The positivity condition for r_2 given by (52) leads to

$$r_2 = \frac{(X_{22}Y_{11} - X_{12}Y_{22})(K'_{x1} - aK'_{y1})}{aX_{11}K'_{y1} - Y_{11}K'_{x1}} > 0 \quad (56)$$

Again by induction, we can show that $X_{22}Y_{11} - X_{12}Y_{22} > 0$ (see Appendix 1) and hence

$$0 < K'_{x1}, \quad \frac{1}{a} \frac{Y_{11}}{X_{11}} < \frac{K'_{y1}}{K'_{x1}} < \frac{1}{a} \quad (57)$$

Note that $(Y_{11}/X_{11}) < 1$ (see Appendix 1) so there is no other solution. The intersection of (54) and (57) with regards to

$(Y_{22}/X_{12}) < (Y_{11}/X_{11})$ and $(Y_{22}/X_{12}) < 1$ (see Appendix 1) leads to

$$0 < K'_{x1} < a, \quad 0 < K'_{y1}, \quad \frac{1}{a} \frac{Y_{11}}{X_{11}} < \frac{K'_{y1}}{K'_{x1}} < \frac{1}{a} \quad \square$$

Corollary 2: In the case of infinite prediction horizon and $q_i = 1, i = 1, 2, \dots, P$ in Theorem 3, the maximum feasibility area is the same as (47).

Proof: By the assumption of $P \rightarrow \infty$ and $q_i = 1, i = 1, 2, \dots, P$ in (51), the assertion follows. \square

Remark 3: The maximum area of feasible gains with control horizons one and two are identical, and therefore the achievable performances are the same.

Theorem 4: Consider the MPC problem with two coincidence points $N_1 = k + i, N_2 = k + j$ and $1 \leq i < j \leq P$. Let the control horizon be 2. The desired feasible gains K'_{xd1} and K'_{ydl} satisfy the following inequalities

$$0 < K'_{xd1} < a, \quad 0 < K'_{ydl}, \quad \frac{1}{a} \frac{\bar{Y}_{11}}{\bar{X}_{11}} < \frac{K'_{ydl}}{K'_{xd1}} < \frac{1}{a} \quad (58)$$

and the tuning equations for achieving these gains are

$$\begin{aligned} r_1 &= \frac{(\bar{Y}_{11}\bar{X}_{12} - \bar{X}_{11}\bar{Y}_{22})(a - K'_{xd1})}{a\bar{X}_{12}K'_{ydl} - \bar{Y}_{22}K'_{xd1}}, \\ r_2 &= \frac{(\bar{X}_{22}\bar{Y}_{11} - \bar{X}_{12}\bar{Y}_{22})(K'_{xd1} - aK'_{ydl})}{a\bar{X}_{11}K'_{ydl} - \bar{Y}_{11}K'_{xd1}} \end{aligned} \quad (59)$$

where

$$\begin{aligned} X_{i1} &= 1 + a + \dots + a^{i-1}, \quad X_{i2} = 1 + a + \dots + a^{i-2}, \\ X_{j1} &= 1 + a + \dots + a^{j-1}, \quad X_{j2} = 1 + a + \dots + a^{j-2} \\ \bar{X}_{11} &= X_{i1}^2 + q_j X_{j1}^2, \quad \bar{X}_{12} = X_{i1}X_{j2} + q_j X_{j1}X_{i2}, \\ \bar{X}_{22} &= X_{j2}^2 + q_j X_{i2}^2 \\ \bar{Y}_{11} &= X_{i1} + q_j X_{j1}, \quad \bar{Y}_{22} = X_{j2} + q_j X_{i2} \end{aligned} \quad (60)$$

Proof: The proof is evident from the results of Theorem 3 and the assumptions. \square

Remark 4: By considering large enough j and q_j in Theorem 4, the maximum feasibility area (47) is achieved.

To derive similar results for the control horizons of 3, 4, ..., the mathematical formulas become extremely complicated. However, to study these cases, the following design experiment is performed. For the control horizon of 3, let r_1, r_2, r_3 and P be tunable parameters. Select these weight parameters randomly in the interval [0.001 10000] and the prediction horizon in the interval [3 200]. Then, K'_{x1} and K'_{y1} are calculated via (17). This test is performed several times with different tuning parameters. Fig. 3 shows the achievable area for these gains.

Note that the dashed line shows the maximum area of feasibility in (47). This test is repeated for control horizons of 4 and 5, and similar results are obtained. This clearly shows that increasing the control horizon beyond 2 does not improve the performance of the MPC for FOPDT models. Referring to note 1, it is noted that these results holds for higher order plants described by FOPDT models.

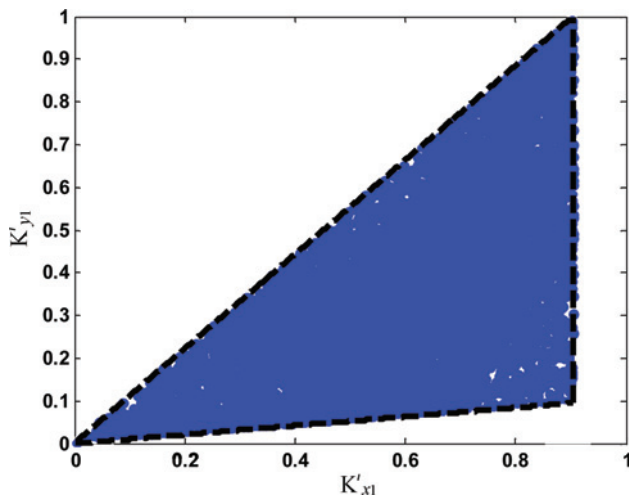


Fig. 3 Maximum feasibility area

3.3 Tuning algorithm

In this section, the above theoretical results are summarised in a tuning algorithm for practical applications. The algorithm is as follows:

Step 1: Derive a proper FOPDT model of the plant in the form of (2).

Step 2: If the model has enough accuracy, then go to step 4 else go to the next step.

Step 3: Find the appropriate gains K'_{x1} and K'_{y1} according to (25). Go to Step 5.

Step 4: Find the appropriate gains K'_{x1} and K'_{y1} according to (16).

Step 5: Test the feasibility of these gains in the maximum feasible area (47). If it is satisfied then go to the next step, if not go back to steps 3 or 4 depending on the accuracy of the FOPDT model and choose another proper desired gains and repeat these steps. If there are no desired gains that satisfy the maximum feasibility conditions, the desired performance is not achievable with the proposed method.

Step 6: Test the control horizon of one (Theorem 1 or 2). Find the proper prediction horizon P such that (41) or (45) is satisfied. If no such P exists go to next step, otherwise use (40) or (44). The procedure is completed.

Step 7: Test the control horizon of two (Theorem 3 or 4). Find the proper prediction horizon P such that (51) or (58) is satisfied. Use (52) or (59). The procedure is terminated.

4 Simulation results

In this section, two examples are presented to show the effectiveness of the proposed tuning methods. In the first example, an FOPDT plant is considered and in the second example, a high order plant is employed to verify the theorems and the tuning algorithm.

4.1 FOPDT plant

Consider the following FOPDT plant [5]

$$G_p(s) = G_m(s) = \frac{1.5e^{-4s}}{10s + 1}$$

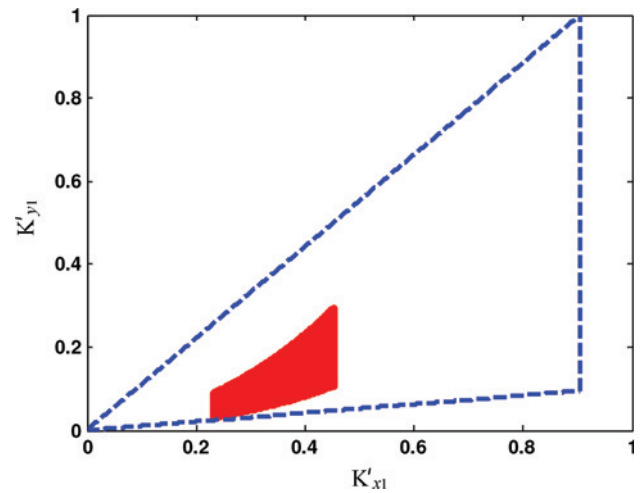


Fig. 4 Desired and feasibility regions of the gains

Let $T_s = 1$ s, the corresponding discrete transfer function is

$$G_p(z^{-1}) = G_m(z^{-1}) = \frac{0.1425z^{-5}}{1 - 0.905z^{-1}}$$

The desired closed-loop performance has a maximum overshoot of 10% and a settling time of 14 s up to 24 s and the corresponding transfer function is

$$G_d(s) = \frac{\omega_n^2 e^{-4s}}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $0.6 \leq \xi \leq 1$ and $0.2 \leq \xi\omega_n \leq 0.4$. Comparing $G_d(s)$ and the closed-loop transfer function (16), the desired gains lie in the region shown in Fig. 4. It is obvious that the desired gains are in the feasible region and the desired performance is achievable. Hence, select $K'_{xd1} = 0.3$ and $K'_{yd1} = 0.1$. Let the input and output constraints be $|u(n)| \leq 1.4$ and $|y(n)| \leq 1.5$, respectively.

Considering the conditions of Theorem 1, (41) gives

$$4 \leq P \leq 6$$

Let $P = 5$, then $N_1 = 5$, $N_2 = 9$. The tuning equations according to (42) are as follows

$$q_P = 1.8$$

$$r = 110.55 \xrightarrow{(6)} R = k_p^2(1 - a)^2 = 2.25$$

Now according to Theorem 2 and (45), we have

$$P \geq 4$$

Let $P = 5$, hence we have $N_1 = 5$, $N_2 = 9$. The tuning equations according to (46) are as follows

$$q_P = 0.68$$

$$r = 25.53 \xrightarrow{(6)} R = k_p^2(1 - a)^2 = 0.52$$

Now the control horizon of two is considered. Let

$$Q = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 & q_P \end{bmatrix}$$

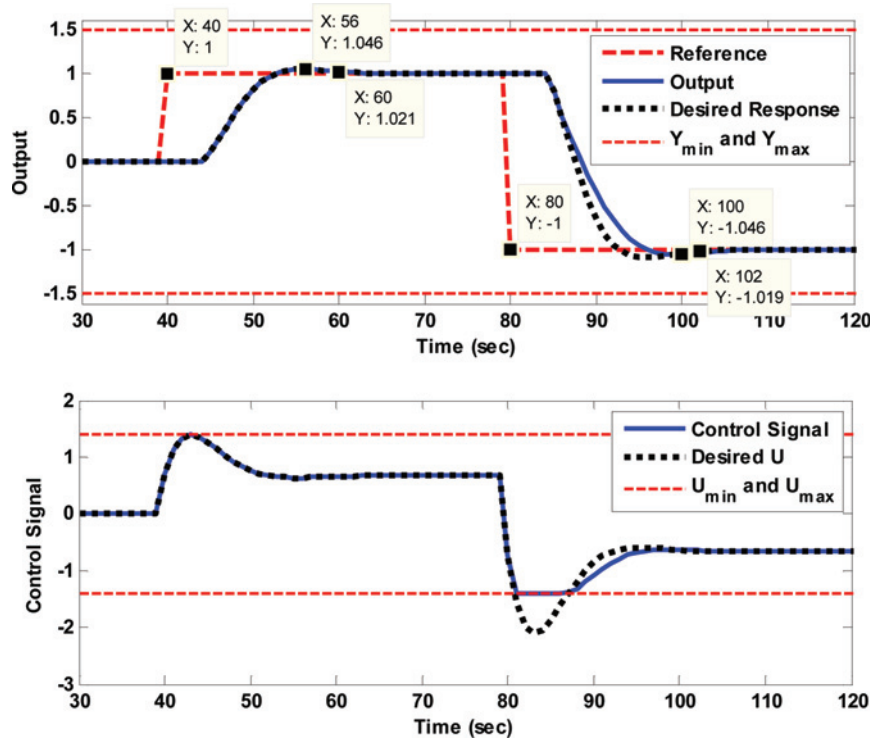


Fig. 5 Closed-loop responses, dotted line: desired response

According to (51), a proper choice can be $P = 5$ and $q_p = 3$, and hence $N_1 = 5$, $N_2 = 9$. Tuning weights according to (52) are

$$r_1 = 75.24, r_2 = 45.02$$

$$\xrightarrow{(6)} \mathbf{R} = k_p^2(1-a)^2 \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} 1.53 & 0 \\ 0 & 0.91 \end{bmatrix}$$

In Theorem 4, the proper choices using (58) are $P = 5$ and $q_p = 3$, and hence $N_1 = 5$, $N_2 = 9$. Also, (59) leads to

$$r_1 = 25.53, r_2 = 10.6$$

$$\xrightarrow{(6)} \mathbf{R} = k_p^2(1-a)^2 \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} 0.52 & 0 \\ 0 & 0.22 \end{bmatrix}$$

Fig. 5 shows the closed-loop step response tracking results with and without active constraints. Note that all of the above tuning equations lead to the response dedicated in Fig. 5.

4.2 Higher order plant

In this example, a higher order plant is considered [8]

$$G_p(s) = \frac{e^{-50s}}{(150s+1)(25s+1)}$$

A proper FOPDT model approximation of the above transfer function is

$$G_m(s) = \frac{e^{-70s}}{157s+1}$$

Let $T_s = 14$ s, so the corresponding discrete time transfer functions are

$$G_m(z^{-1}) = \frac{0.085z^{-6}}{1-0.915z^{-1}}$$

and

$$G_p(z^{-1}) = z^{-4} \frac{0.00438 + 0.0288z^{-1} + 0.00504z^{-2}}{1 - 1.482z^{-1} + 0.5203z^{-2}}$$

Hence, the closed-loop transfer function (25) is

$$G_{cl}(z) = \frac{K'_{yd1}z^2(0.0044z^3 + 0.0248z^2 - 0.0213z - 0.0046)}{\Delta_{cl}(z)}$$

where

$$\begin{aligned} \Delta_{cl}(z) = & z^9(0.0853) \\ & + z^8(-0.2897 + 0.0853(K'_{xd1} + K'_{yd1})) \\ & + z^7(0.3645 - 0.2117K'_{xd1} - 0.1264K'_{yd1}) \\ & + z^6(-0.2006 + 0.1708K'_{xd1} + 0.0444K'_{yd1}) \\ & + z^5(0.0406 - 0.0400K'_{xd1} + 0.0044K'_{yd1}) \\ & + z^4(0.0204K'_{xd1} + 0.0248K'_{yd1}) \\ & + z^3(-0.1314K'_{xd1} - 0.1066K'_{yd1}) \\ & + z^2(0.2284K'_{xd1} + 0.1218K'_{yd1}) \\ & + z(-0.1662K'_{xd1} - 0.0444K'_{yd1}) + 0.0444K'_{xd1} \end{aligned}$$

The stability region of the closed-loop characteristic polynomial is as shown in Fig. 6.

In the next step, the desired performance is considered. The desired performance has no overshoot and a settling time of less than 400 s. Simulation results are used to find the proper gains K'_{x1} and K'_{y1} that satisfy the desired performance. The desired gains that yield the desired performance are illustrated in Fig. 7. Note that the desired performance is in the feasible area and, therefore, a proper tuning set exists to ensure closed-loop stability and performance. Consider the input constraint $|u(n)| \leq 2.3$.

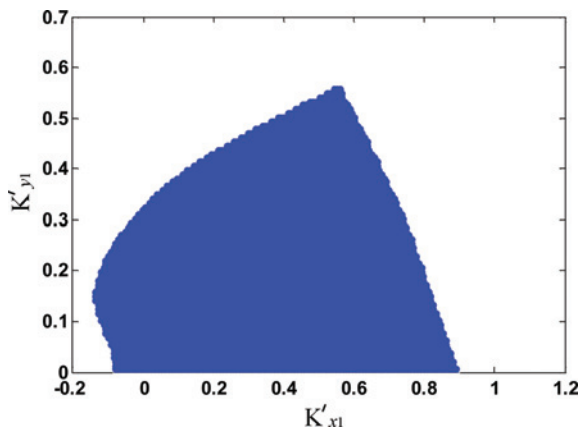


Fig. 6 Stability region

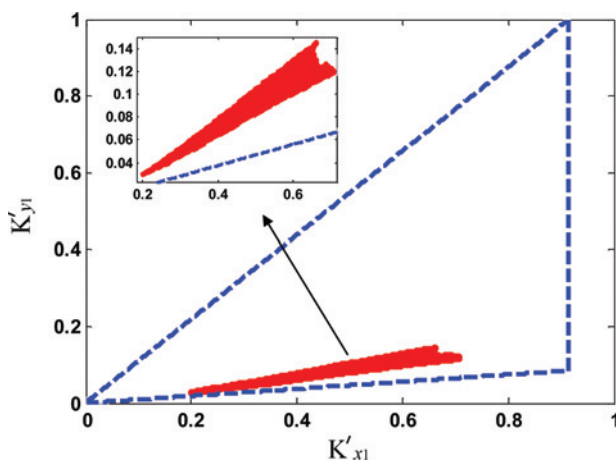


Fig. 7 Desired and feasibility areas of the gains

Using Fig. 7, we choose $K'_{xd1} = 0.25$ and $K'_{yd1} = 0.04$. To obtain these gains, let $q_i = 1, i = 1 : P$. The proper P should be obtained according to (51), which gives

$$P \geq 17$$

So $P = 17$ can be a good choice, and hence $N_1 = 6, N_2 = 22$. Employing (52) gives

$$r_1 = 1075.8, \quad r_2 = 828.1$$

$$\xrightarrow{(6)} R = k_p^2(1-a)^2 \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} 7.83 & 0 \\ 0 & 6.03 \end{bmatrix}$$

Fig. 8 shows the closed-loop step responses. Note that, in the case of no active constraints the output has no overshoot and the settling time is about 350 s.

5 Conclusions

A tuning strategy is developed for single-input single-output MPC of FOPDT models when constraints are inactive. The closed-loop transfer functions are obtained and subsequently, the stability and achievable performances are addressed. Tuning formulas to attain the desired performances are derived. It is shown that for FOPDT plants, the best achievable performance can be accomplished with a control horizon of two, which greatly simplify the controller structure. Plants of higher order modelled by a FOPDT transfer function are also considered. Finally, simulation results are used to show the effectiveness of the proposed tuning method.

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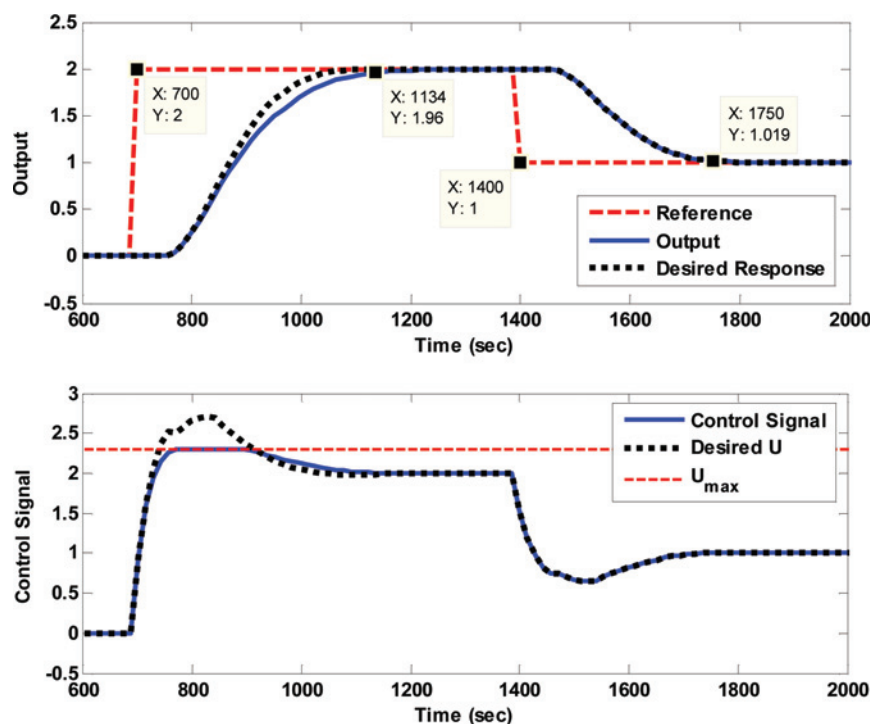


Fig. 8 Closed-loop responses, dotted line: desired response

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8 Appendix 1

Several mathematical manipulations on (33) lead to (35). We have

$$\begin{aligned} \sum_{j=1}^{N-1} (g_{i+j} - g_j) z^{-j} &= \sum_{j=1}^{N-1} g_{i+j} z^{-j} - \sum_{j=k+1}^{N-1} g_j z^{-j} \\ &= k_p \left\{ \sum_{j=1}^{N-1} z^{-j} - \sum_{j=k+1}^{N-1} z^{-j} - \sum_{j=1}^{N-1} a^{i+j-k} z^{-j} + \sum_{j=k+1}^{N-1} a^{j-k} z^{-j} \right\} \\ &= k_p \left\{ \sum_{j=1}^k z^{-j} - \sum_{j=1}^{N-1} a^{i+j-k} z^{-j} + \sum_{j=k+1}^{N-1} a^{j-k} z^{-j} \right\} \end{aligned}$$

For large enough model horizon N , this gives

$$\begin{aligned} (1 - az^{-1})(1 - z^{-1}) \sum_{j=1}^{N-1} (g_{i+j} - g_j) z^{-j} \\ &= k_p (1 - az^{-1})(1 - z^{-1}) \left(\sum_{j=1}^k z^{-j} - \sum_{j=1}^{N-1} a^{i+j-k} z^{-j} \right. \\ &\quad \left. + \sum_{j=k+1}^{N-1} a^{j-k} z^{-j} \right) = k_p ((1 - az^{-1})(z^{-1} - z^{-k-1}) \\ &\quad + (1 - z^{-1})(-a^{i+1-k} z^{-1} + az^{-k-1})) \\ &= k_p (z^{-1} - az^{-2} - (1 - a)z^{-k-1} \\ &\quad + (1 - z^{-1})z^{-1}(-a^{i+1-k} + az^{-k})) \end{aligned}$$

hence

$$\begin{aligned} \Delta_{cl}(z^{-1}) &= k_p (z^{-1} - az^{-2} - (1 - a)z^{-k-1}) \sum_{i=k+1}^{k+p} c_i \\ &\quad - k_p (1 - z^{-1}) z^{-1} a^{1-k} \sum_{i=k+1}^{k+p} c_i a^i \\ &\quad + k_p (1 - a) z^{-k-1} \sum_{i=k+1}^{k+p} c_i \\ &= 1 + z^{-1} \left(-1 - a + k_p \sum_{i=k+1}^{k+p} c_i - k_p a^{1-k} \sum_{i=k+1}^{k+p} c_i a^i \right) \\ &\quad + z^{-2} \left(a - ak_p \sum_{i=k+1}^{k+p} c_i + k_p a^{1-k} \sum_{i=k+1}^{k+p} c_i a^i \right) \end{aligned}$$

To prove the inequalities and equalities of Section 3, let

$$\begin{aligned} X_{11}(P) &= 1 + q_2(1 + a)^2 + q_3(1 + a + a^2)^2 + \dots \\ &\quad + q_P(1 + a + \dots + a^{P-1})^2 \\ X_{12}(P) &= q_2(1 + a) + q_3(1 + a)(1 + a + a^2) + \dots \\ &\quad + q_P(1 + \dots + a^{P-2})(1 + \dots + a^{P-1}) \\ X_{22}(P) &= q_2 + q_3(1 + a)^2 + q_4(1 + a + a^2)^2 + \dots \\ &\quad + q_P(1 + a + \dots + a^{P-2})^2 \\ Y_{11}(P) &= 1 + q_2(1 + a) + q_3(1 + a + a^2) + \dots \\ &\quad + q_P(1 + a + \dots + a^{P-1}) \\ Y_{22}(P) &= q_2 + q_3(1 + a) + q_4(1 + a + a^2) + \dots \\ &\quad + q_P(1 + \dots + a^{P-2}) \end{aligned} \quad (61)$$

The proof of all inequalities and equalities are by induction, showing that the truth of inequality or equality for $P = n$ leads to the truth of $P = n + 1$. According to the control horizon of 2, the minimum prediction horizon is 2 and the initial point is $P = 2$.

Proof of $X_{11} > Y_{11}$. For $P = 2$

$$X_{11}(2) = 1 + q_2(1 + a)^2 > Y_{11}(2) = 1 + q_2(1 + a)$$

For $P = n$, we have

$$X_{11}(n) > Y_{11}(n)$$

For $P = n + 1$

$$\begin{aligned} X_{11}(n+1) &= X_{11}(n) + q_{n+1}(1 + a + \dots + a^n)^2 \\ Y_{11}(n+1) &= Y_{11}(n) + q_{n+1}(1 + a + \dots + a^n) \end{aligned}$$

hence

$$(1 + a + \dots + a^n) > 1$$

So $X_{11}(n+1) > Y_{11}(n+1)$ and the proof is completed. \square

Proof of $X_{12} > Y_{22}$: For $P = 2$

$$X_{12}(2) = q_2(1 + a) > Y_{22}(2) = q_2$$

For $P = n$, we have

$$X_{12}(n) > Y_{22}(n)$$

For $P = n + 1$

$$\begin{aligned} X_{12}(n+1) &= X_{12}(n) + q_{n+1}(1 + \dots + a^{n-1})(1 + \dots + a^n) \\ Y_{22}(n+1) &= Y_{22}(n) + q_{n+1}(1 + a + \dots + a^{n-1}) \end{aligned}$$

hence

$$\begin{aligned} (1 + a + \dots + a^{n-1})(1 + a + \dots + a^n) \\ > (1 + a + \dots + a^{n-1}) \end{aligned}$$

So $X_{12}(n+1) > Y_{22}(n+1)$ and the proof is completed. \square

Proof of $X_{22}Y_{11} - X_{12}Y_{22} > 0$: For $P = 2$

$$\begin{aligned} X_{22}(2)Y_{11}(2) - X_{12}(2)Y_{22}(2) &= q_2(1 + q_2(1 + a)) \\ &\quad - q_2(1 + a)q_2 = q_2 > 0 \end{aligned}$$

For $P = n$, we have

$$X_{22}(n)Y_{11}(n) - X_{12}(n)Y_{22}(n) > 0$$

For $P = n + 1$

$$\begin{aligned} X_{22}(n+1)Y_{11}(n+1) - X_{12}(n+1)Y_{22}(n+1) \\ &= X_{22}(n)Y_{11}(n) - X_{12}(n)Y_{22}(n) \\ &\quad + q_{n+1} \{ Y_{11}(n)(1 + a + \dots + a^{n-1})^2 \\ &\quad + X_{22}(n)(1 + a + \dots + a^n) \\ &\quad - Y_{22}(n)(1 + \dots + a^{n-1})(1 + \dots + a^n) \\ &\quad - X_{12}(n)(1 + a + \dots + a^{n-1}) \} \\ &= X_{22}(n)Y_{11}(n) - X_{12}(n)Y_{22}(n) \\ &\quad + q_{n+1} \left\{ \sum_{j=0}^{n-1} (a^j)^2 + \sum_{i=2}^n q_i a^{2(i-1)} \sum_{j=0}^{n-i} (a^j)^2 \right\} \end{aligned} \quad (62)$$

Note that $\sum_{j=0}^{n-1} (a^j)^2 + \sum_{i=2}^n q_i a^{2(i-1)} \sum_{j=0}^{n-i} (a^j)^2 > 0$, so the assertion follows. \square

Proof of $X_{12}Y_{11} - X_{11}Y_{22} > 0$: For $P = 2$

$$\begin{aligned} X_{12}(2)Y_{11}(2) - X_{11}(2)Y_{22}(2) \\ &= q_2(1 + a)(1 + q_2(1 + a)) \\ &\quad - (1 + q_2(1 + a))^2 q_2 = q_2 a > 0 \end{aligned}$$

For $P = n$, we have

$$X_{12}(n)Y_{11}(n) - X_{11}(n)Y_{22}(n) > 0$$

For $P = n + 1$

$$\begin{aligned} X_{12}(n+1)Y_{11}(n+1) - X_{11}(n+1)Y_{22}(n+1) \\ &= X_{12}(n)Y_{11}(n) - X_{11}(n)Y_{22}(n) \\ &\quad + q_{n+1} \{ Y_{11}(n)(1 + \dots + a^{n-1})(1 + \dots + a^n) \\ &\quad + X_{12}(n)(1 + \dots + a^n) - Y_{22}(n)(1 + \dots + a^n)^2 \\ &\quad - X_{11}(n)(1 + \dots + a^{n-1}) \} \\ &= X_{12}(n)Y_{11}(n) - X_{11}(n)Y_{22}(n) \\ &\quad + q_{n+1} a \left\{ \sum_{j=0}^{n-1} (a^j)^2 + \sum_{i=2}^n q_i a^{2(i-1)} \sum_{j=0}^{n-i} (a^j)^2 \right\} \end{aligned}$$

Note that $a \sum_{j=0}^{n-1} (a^j)^2 + a \sum_{i=2}^n q_i a^{2(i-1)} \sum_{j=0}^{n-i} (a^j)^2 > 0$, so the assertion follows. \square

Proof of $X_{22}Y_{11} - X_{12}Y_{22} = X_{11}X_{22} - X_{12}^2$: For $P = 2$

$$\begin{aligned} X_{22}(2)Y_{11}(2) - X_{12}(2)Y_{22}(2) \\ &= q_2 + q_2^2(1 + a) - q_2^2(1 + a) \\ &= q_2X_{11}(2)X_{22}(2) - X_{12}^2(2) \\ &= q_2 + q_2^2(1 + a)^2 - q_2^2(1 + a)^2 = q_2 \end{aligned}$$

For $P = n$, we have

$$X_{22}(n)Y_{11}(n) - X_{12}(n)Y_{22}(n) = X_{11}(n)X_{22}(n) - X_{12}^2(n)$$

For $P = n + 1$

$$\begin{aligned} X_{11}(n+1)X_{22}(n+1) - X_{12}^2(n+1) \\ &= X_{11}(n)X_{22}(n) - X_{12}^2(n) \\ &\quad + q_{n+1} \{ X_{22}(n)(1 + a + \dots + a^n)^2 \\ &\quad + X_{11}(n)(1 + a + \dots + a^{n-1})^2 \\ &\quad - 2X_{12}(n)(1 + \dots + a^{n-1})(1 + \dots + a^n) \} \\ &= X_{11}(n)X_{22}(n) - X_{12}^2(n) \\ &\quad + q_{n+1} \left\{ \sum_{j=0}^{n-1} (a^j)^2 + \sum_{i=2}^n q_i a^{2(i-1)} \sum_{j=0}^{n-i} (a^j)^2 \right\} \end{aligned}$$

According to the (62) the statement follows. \square